

NOTES ON YAU'S "RICCI CURVATURE AND MONGE-AMPERE EQUATION"

TSECHING LIEN

1. THE CALABI CONJECTURE

Let M be a compact Kähler manifold of dimension m , with the Kähler metric given by $\sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$, and the Kähler form given by $\omega = \sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then the Ricci curvature of M is given by

$$Ric(\omega) = \sum_{i,j} R_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

where

$$R_{i\bar{j}} = - \frac{\partial^2}{\partial z^i \partial \bar{z}^k} [\log \det(g_{s\bar{t}})].$$

The (1,1) form $Ric(\omega) := (\sqrt{-1}/2\pi) \sum_{i,j} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is closed, and equal to

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} [\log \det(g_{s\bar{t}})].$$

The cohomology class of this (1,1) form is the first Chern class of M .

Thus, a necessary condition for a (1,1) form to be the Ricci form of another Kähler metric in the same class as ω is that it is closed and its cohomology class is the same as the first Chern class, $Ch(M)$, of the manifold. This gives us the Calabi conjecture.

Conjecture. The above necessary conditions are in fact sufficient. That is, if a (1,1) form ρ is closed and its cohomology class $[\rho] = [Ch(M)]$, then

$$\rho = Ric(\tilde{g})$$

for a Kähler metric \tilde{g} .

In fact, we can do better. We can find a Kähler metric in the same cohomology class as our original Kähler metric, ω so that its Ricci form is ρ .

We can interpret the Calabi Conjecture into a problem of solving non-linear partial differential equations in the following way. If the (1,1) form $\rho = (\sqrt{-1}/2\pi) \sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \otimes d\bar{z}^j$ is closed and represents the first Chern class of M , then we can find a smooth function, F such that

$$(1.1) \quad \tilde{R}_{i\bar{j}} - R_{i\bar{j}} = - \frac{\partial^2 F}{\partial z^i \partial \bar{z}^j}$$

Also, if $\sum_{i,j} \tilde{R}_{i\bar{j}} dz^i \otimes d\bar{z}^j$ is the Ricci tensor of some Kähler metric $\sum_{i,j} \tilde{g}_{i\bar{j}} dz^i \otimes d\bar{z}^j$, then we also have that

$$\tilde{R}_{i\bar{j}} = - \frac{\partial^2}{\partial z^i \partial \bar{z}^k} [\log \det(\tilde{g}_{s\bar{t}})]$$

and thus,

$$\begin{aligned} \tilde{R}_{i\bar{j}} - R_{i\bar{j}} &= - \frac{\partial^2}{\partial z^i \partial \bar{z}^k} [\log \det(g_{s\bar{t}})] + \frac{\partial^2}{\partial z^i \partial \bar{z}^k} [\log \det(\tilde{g}_{s\bar{t}})] \\ &= - \frac{\partial^2}{\partial z^i \partial \bar{z}^k} [\log \det(g_{s\bar{t}}) - \log \det(\tilde{g}_{s\bar{t}})] \\ &= - \frac{\partial^2}{\partial z^i \partial \bar{z}^k} \left[\log \left(\frac{\det g_{s\bar{t}}}{\det \tilde{g}_{s\bar{t}}} \right) \right] \\ &= -\partial\bar{\partial}F \end{aligned}$$

Hence,

$$\partial\bar{\partial} \log \left(\frac{\det g_{s\bar{t}}}{\det \tilde{g}_{s\bar{t}}} \right) = \partial\bar{\partial}F$$

By the Maximum Principle, since M is compact and $\det(g_{s\bar{t}})/\det(\tilde{g}_{s\bar{t}})$ is a globally defined function, we must have that $\log \det(g_{s\bar{t}})/\det(\tilde{g}_{s\bar{t}}) - F = C$, for some constant $C > 0$. Thus, raising both sides of the equation to the exponential, we have

$$(1.2) \quad \det(\tilde{g}_{s\bar{t}}) = C \exp\{F\} \det(g_{s\bar{t}}).$$

If the two metrics are cohomologous, then we have that they differ by some smooth function. That is,

$$\tilde{g}_{s\bar{t}} = g_{s\bar{t}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}.$$

So, we have the partial differential equation,

$$(1.3) \quad \det\left(g_{s\bar{t}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}\right) = C \exp\{F\} \det(g_{i\bar{j}}).$$

Conversely, if we can solve 1.3 so that the solution φ is smooth, and $\sum_{i,j} (g_{i\bar{j}} + \partial^2 \varphi / \partial z^i \partial \bar{z}^j) dz^i \otimes d\bar{z}^j$ defines a Kähler metric, then there is only one choice of C so that

$$(1.4) \quad C \int_M \exp\{F\} = \text{Vol}(M).$$

Yau solves (1.3) with the necessary compatibility conditions, (1.4). This is done using the continuity method. Consider the equation,

$$(1.5) \quad \det(g_{s\bar{t}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}) = C \exp\{tF\} \det(g_{i\bar{j}}).$$

Note that we can solve this equation when $t = 0$; in particular, let $\varphi = 0$. Let S be the subset of $t \in [0, 1]$ such that (1.5) can be solved. S is nonempty since it contains 0. Yau then shows that S is both open and closed as a set, and gets that S must equal $[0, 1]$. Hence, we can solve the equation 1.5 when $t = 1$.

2. S IS CLOSED.

We will need some estimates.

2.1. Second Order estimates. Suppose φ solves 1.3. By the Schauder estimates, we have

$$(2.1) \quad \|\varphi\|_{C^{3,\alpha}} \leq C(\omega) (\|\Delta\varphi\|_{C^{1,\alpha}} + \|\varphi\|_{C^0})$$

$$(2.2) \quad \|\Delta\varphi\|_{C^{1,\alpha}} \leq C(\|\varphi_{i\bar{j}}\|_{C^\alpha}) (\|\Delta_\varphi \Delta\varphi\|_{C^0} + \|\Delta\varphi\|_{C^0})$$

We have these a priori estimates, so we need to bound

- a.) $\|\Delta\varphi\|_{C^0}$
- b.) $\|\varphi\|_{C^0}$
- c.) $\|\Delta_\varphi \Delta\varphi\|_{C^0}$

We start with the last one. Let $\Delta' = \Delta_\varphi$, the Laplacian with the metric $g'_{i,\bar{j}} := g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$. We also denote the entries of the inverse matrix of the metric with upper indices, i.e. by $g'^{i\bar{j}}$. That is,

$$\sum_k g'^{k\bar{i}} g'_{k\bar{j}} = \delta_{\bar{j}}^i$$

Then,

$$\begin{aligned}
(2.3) \quad \Delta'(\Delta\varphi) &= \sum_{k,l} g'^{k\bar{l}} \frac{\partial^2}{\partial z^k \partial \bar{z}^l} \left(\sum_{i,j} g^{i\bar{j}} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) \\
&= \sum_{k,l} g'^{k\bar{l}} \frac{\partial}{\partial z^k} \left(\sum_{i,j} \frac{\partial g^{i\bar{j}}}{\partial z^l} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} + g^{i\bar{j}} \frac{\partial^3 \varphi}{\partial \bar{z}^l \partial z^i \partial \bar{z}^j} \right) \\
&= \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} + \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^l} \frac{\partial^3 \varphi}{\partial z^k \partial z^i \partial \bar{z}^j} \\
&\quad + \sum_{i,j,k,l} \frac{\partial g^{i\bar{j}}}{\partial z^k} \frac{\partial^3 \varphi}{\partial \bar{z}^l \partial z^i \partial \bar{z}^j} + \sum_{i,j,k,l} g'^{k\bar{l}} g^{i\bar{j}} \frac{\partial^4 \varphi}{\partial z^k \partial \bar{z}^l \partial z^i \partial \bar{z}^j}
\end{aligned}$$

At a point with normal coordinates, we have $g_{i\bar{j}} = \delta_{ij}$; $\partial g_{i\bar{j}}/\partial z^k = \partial g_{i\bar{j}}/\partial \bar{z}^l = 0$, which also gives us that $g^{i\bar{j}} = \delta_{i\bar{j}}$; $\partial g^{i\bar{j}}/\partial z^k = \partial g^{i\bar{j}}/\partial \bar{z}^l = 0$ (See Appendix A).

Since the Laplacian is a globally defined operator, it does not depend on the choice of local coordinates, and we can pick our coordinates so that they are normal and $\partial^2 \varphi/\partial z^k \partial \bar{z}^l =: \varphi_{k\bar{l}} = \delta_{k\bar{l}} \varphi_{k\bar{k}}$. This gives us

$$(2.4) \quad g'^{k\bar{l}} = \delta_{kl}(1 + \varphi_{k\bar{l}})^{-1}$$

Thus, at normal coordinates, we lose the second and third terms, and we have

$$\begin{aligned}
\Delta'(\Delta\varphi) &= \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{i,j,k,l} g'^{k\bar{l}} \frac{\partial^4 \varphi}{\partial z^k \partial \bar{z}^l \partial z^i \partial \bar{z}^j} \\
&= \sum_{i,j,k} \frac{1}{(1 + \varphi_{k\bar{k}})} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^k} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} + \sum_{i,k} \frac{1}{(1 + \varphi_{k\bar{k}})} \frac{\partial^4 \varphi}{\partial z^k \partial \bar{z}^l \partial z^i \partial \bar{z}^j}
\end{aligned}$$

The above equation looks like a fourth order equation, which would be unfortunate for we don't want to be bounding more and more derivatives of φ , but in fact, it is only 3rd order because φ solves (1.3), and we can replace the fourth order term by a term that does not depend on φ .

Consider

$$(2.5) \quad F = \log \det \left(g^{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) - \log \det \left(g^{i\bar{j}} \right),$$

we differentiate (2.5) first with respect to z^k and then \bar{z}^l . See Appendix B for differentiation of the logarithm of a determinant.

$$\begin{aligned}
 \frac{\partial F}{\partial z^k} &= \sum_{i,\bar{j}} g^{i\bar{j}} \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k} \right) - \sum_{i,\bar{j}} g^{i\bar{j}} \frac{\partial g^{i\bar{j}}}{\partial z^k} \\
 \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l} &= \sum_{i,\bar{j}} g^{i\bar{j}} \left(\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k \partial \bar{z}^l} \right) \\
 &\quad + \sum_{i,\bar{j}} \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^l} \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k} \right) \\
 &\quad - \sum_{i,\bar{j}} \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^l} \frac{\partial g_{i\bar{j}}}{\partial z^k} - \sum_{i,\bar{j}} g^{i\bar{j}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^l}
 \end{aligned}
 \tag{2.6}$$

Note that

$$\begin{aligned}
 \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^l} &= - \sum_{t,n} g^{i\bar{n}} \frac{\partial g'_{t\bar{n}}}{\partial \bar{z}^l} g^{t\bar{j}} \\
 &= - \sum_{t,n} g^{i\bar{n}} \left(\frac{\partial g_{t\bar{n}}}{\partial \bar{z}^l} + \frac{\partial^3 \varphi}{\partial \bar{z}^l \partial z^t \partial \bar{z}^n} \right) g^{t\bar{j}}
 \end{aligned}
 \tag{2.7}$$

Thus, again at normal coordinates, we have

$$\begin{aligned}
 \Delta F &= \sum_{k,\bar{l}} g^{k\bar{l}} \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l} \\
 &= \sum_{i,j,k} \frac{1}{(1 + \varphi_{i\bar{i}})} \frac{\partial^2 g_{i\bar{i}}}{\partial z^k \partial \bar{z}^k} + \sum_{i,k} \frac{1}{(1 + \varphi_{i\bar{i}})} \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^i \partial z^k \partial \bar{z}^k} \\
 &\quad - \sum_{i,j,k} \frac{1}{(1 + \varphi_{j\bar{j}})} \frac{1}{(1 + \varphi_{i\bar{i}})} \left(\frac{\partial^3 \varphi}{\partial \bar{z}^i \partial z^j \partial \bar{z}^k} \right) \left(\frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k} \right) \\
 &\quad - \sum_{i,k} \frac{\partial^2 g_{i\bar{i}}}{\partial z^k \partial \bar{z}^k}
 \end{aligned}
 \tag{2.8}$$

So,

$$\begin{aligned}
 &\sum_{i,k} \frac{1}{(1 + \varphi_{i\bar{i}})} \frac{\partial^4 \varphi}{\partial z^i \partial \bar{z}^i \partial z^k \partial \bar{z}^k} \\
 &= \Delta F + \sum_{i,j,k} \frac{1}{(1 + \varphi_{j\bar{j}})} \frac{1}{(1 + \varphi_{i\bar{i}})} \varphi_{i\bar{j}k} \varphi_{i\bar{j}k} \\
 &\quad - \sum_{i,k} \frac{1}{(1 + \varphi_{i\bar{i}})} \frac{\partial^2 g_{i\bar{i}}}{\partial z^k \partial \bar{z}^k} - \sum_{i,k} \frac{\partial^2 g_{i\bar{i}}}{\partial z^k \partial \bar{z}^k}
 \end{aligned}
 \tag{2.9}$$

$$\begin{aligned}
\Delta'(\Delta\varphi) &= \Delta F + \sum_{i,j,k} \frac{1}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \varphi_{i\bar{j}k} \varphi_{i\bar{j}\bar{k}} \\
&\quad - \sum_{i,k} \frac{1}{(1+\varphi_{i\bar{i}})} \frac{\partial^2 g_{i\bar{i}}}{\partial z^k \partial \bar{z}^k} + \sum_{i,k} \frac{\partial^2 g_{i\bar{i}}}{\partial z^k \partial \bar{z}^k} \\
(2.10) \quad &\quad + \sum_{i,j,k} \frac{1}{(1+\varphi_{k\bar{k}})} \left(\frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^k} \right) (\varphi_{i\bar{j}})
\end{aligned}$$

The coefficients of the Ricci tensor are given at normal coordinates by

$$R_{i\bar{j}n\bar{m}} = - \frac{\partial^2 g_{i\bar{j}}}{\partial z^n \partial \bar{z}^m}.$$

More details on the Ricci tensor is given in Appendix C. Thus, we have

$$\begin{aligned}
\Delta'(\Delta\varphi) &= \Delta F + \sum_{i,j,k} \frac{1}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \varphi_{i\bar{j}k} \varphi_{i\bar{j}\bar{k}} \\
(2.11) \quad &\quad + \sum_{i,k} \frac{1}{(1+\varphi_{i\bar{i}})} R_{i\bar{i}k\bar{k}} - \sum_{i,k} R_{i\bar{i}k\bar{k}} + \sum_{i,j,k} \frac{1}{(1+\varphi_{k\bar{k}})} \varphi_{i\bar{j}} R_{i\bar{j}k\bar{k}}
\end{aligned}$$

Let's ignore the first two terms and just consider the last three terms.

$$\begin{aligned}
&\sum_{i,k} \frac{1}{(1+\varphi_{i\bar{i}})} R_{i\bar{i}k\bar{k}} - \sum_{i,k} R_{i\bar{i}k\bar{k}} + \sum_{i,j,k} \frac{1}{(1+\varphi_{k\bar{k}})} \varphi_{i\bar{j}} R_{i\bar{j}k\bar{k}} \\
&= \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{-\varphi_{i\bar{i}}}{(1+\varphi_{i\bar{i}})} + \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\varphi_{i\bar{i}}}{(1+\varphi_{k\bar{k}})} \\
&= \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{-\varphi_{i\bar{i}}(1+\varphi_{k\bar{k}}) + \varphi_{i\bar{i}}(1+\varphi_{i\bar{i}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \\
&= \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\varphi_{i\bar{i}}(\varphi_{i\bar{i}} - \varphi_{k\bar{k}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \\
(2.12) \quad &= \frac{1}{2} \left(\sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\varphi_{i\bar{i}}(\varphi_{i\bar{i}} - \varphi_{k\bar{k}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \right. \\
&\quad \left. + \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\varphi_{k\bar{k}}(\varphi_{k\bar{k}} - \varphi_{i\bar{i}})}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \right) \\
&= \frac{1}{2} \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{(\varphi_{i\bar{i}} - \varphi_{k\bar{k}})^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})}
\end{aligned}$$

$$\begin{aligned}
 &\geq \left(\inf_{i \neq l} R_{i\bar{i}k\bar{k}} \right) \left(\frac{1}{2} \sum_{i,k} \frac{((1 + \varphi_{i\bar{i}}) - (1 + \varphi_{k\bar{k}}))^2}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} \right) \\
 &= \left(\inf_{i \neq l} R_{i\bar{i}k\bar{k}} \right) \left(\frac{1}{2} \left(\sum_{i,k} \frac{(1 + \varphi_{i\bar{i}})^2}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} - \sum_{i,k} \frac{2(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} \right. \right. \\
 &\quad \left. \left. + \sum_{i,k} \frac{(1 + \varphi_{k\bar{k}})^2}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} \right) \right) \\
 &= \left(\inf_{i \neq k} R_{i\bar{i}k\bar{k}} \right) \left(\sum_{i,k} \left(\frac{(1 + \varphi_{i\bar{i}})}{(1 + \varphi_{k\bar{k}})} - 1 \right) \right) \\
 &= \left(\inf_{i \neq k} R_{i\bar{i}k\bar{k}} \right) \left(\sum_{i,k} \frac{(1 + \varphi_{i\bar{i}})}{(1 + \varphi_{k\bar{k}})} - m^2 \right)
 \end{aligned}$$

Notice that the fourth equality follows because the previous sum is symmetric in the indices since $R_{i\bar{i}k\bar{k}}$ is symmetric in the first and third indices, and the second and fourth indices. Combining (2.12) with (2.11), we have

$$\begin{aligned}
 \Delta'(\Delta\varphi) &\geq \Delta F + \sum_{i,j,k} \frac{1}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} \varphi_{i\bar{j}k} \varphi_{i\bar{j}k} \\
 (2.13) \quad &+ \left(\inf_{i \neq k} R_{i\bar{i}k\bar{k}} \right) \left[\sum_{i,k} \frac{1 + \varphi_{i\bar{i}}}{1 + \varphi_{k\bar{k}}} - m^2 \right]
 \end{aligned}$$

(2.14)

Notice the following identity

$$(2.15) \quad \left(\sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \right)^{m-1} = ((1 + \varphi_{1\bar{1}}) + \dots + (1 + \varphi_{m\bar{m}}))^{m-1},$$

which by the "multinomial theorem" gives that

$$\begin{aligned}
 \left(\sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \right)^{m-1} &\geq \sum_i \frac{1}{\prod_{j \neq i} (1 + \varphi_{j\bar{j}})} \\
 &= \left(\frac{\sum_i (1 + \varphi_{i\bar{i}})}{\prod_i (1 + \varphi_{i\bar{i}})} \right)
 \end{aligned}$$

Thus, we have the identity

$$(2.16) \quad \sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \geq \left(\frac{\sum_i (1 + \varphi_{i\bar{i}})}{\prod_i (1 + \varphi_{i\bar{i}})} \right)^{\frac{1}{m-1}}.$$

$$= (m + \Delta\varphi)^{1/(m-1)} \left(\frac{1}{\prod_i (1 + \varphi_{i\bar{i}})} \right)^{\frac{1}{m-1}}$$

We have that φ solves (1.3), so at normal coordinates,

$$\exp\{F\} = \prod_i (1 + \varphi_{i\bar{i}})$$

$$\left(\frac{1}{\exp\{F\}} \right)^{\frac{1}{m-1}} = \left(\frac{1}{\prod_i (1 + \varphi_{i\bar{i}})} \right)^{\frac{1}{m-1}}$$

$$(2.17) \quad \Rightarrow \quad \exp\left\{ \frac{-F}{m-1} \right\} = \left(\frac{1}{\prod_i (1 + \varphi_{i\bar{i}})} \right)^{\frac{1}{m-1}}$$

Now, $\sum \frac{1}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \varphi_{i\bar{j}k} \varphi_{i\bar{j}k}$ is positive because each $(1 + \varphi_{i\bar{i}})$ is the eigenvalue of the positive definite Hermitian matrix, $g^{i\bar{j}}$. If we knew $\inf_{i \neq k} R_{i\bar{i}k\bar{k}}$ were positive enough, that is, > 1 , then we could ignore that second term of (2.13), and by (2.16) and (2.17), we would have a bound on $(m + \Delta\varphi)$ since

$$\sum_{i,k} \frac{1 + \varphi_{i\bar{i}}}{1 + \varphi_{k\bar{k}}} = (m + \Delta\varphi) \left(\sum_k \frac{1}{1 + \varphi_{k\bar{k}}} \right)$$

Unfortunately, we don't know the sign of $\inf_{i \neq k} R_{i\bar{i}k\bar{k}}$. So instead, for some positive constant, C , we consider $\Delta'(\exp\{-C\varphi\}(m + \Delta\varphi))$.

$$\begin{aligned} \Delta'(\exp\{-C\varphi\}(m + \Delta\varphi)) &= \sum_{k,l} g^{k\bar{l}} \left(\frac{\partial^2}{\partial z^k \partial \bar{z}^l} (\exp\{-C\varphi\}(m + \Delta\varphi)) \right) \\ &= \sum_{k,l} g^{k\bar{l}} \frac{\partial}{\partial z^k} \left(-C \exp\{-C\varphi\} \varphi_{\bar{l}}(m + \Delta\varphi) \right. \\ &\quad \left. + \exp\{-C\varphi\} (\Delta\varphi)_{\bar{l}} \right) \\ &= \sum_{k,l} g^{k\bar{l}} \left(C^2 \exp\{-C\varphi\} \varphi_k \varphi_{\bar{l}}(m + \Delta\varphi) \right. \\ &\quad - C \exp\{-C\varphi\} \varphi_{k\bar{l}}(m + \Delta\varphi) \\ &\quad - C \exp\{-C\varphi\} \varphi_k (\Delta\varphi)_{\bar{l}} \\ &\quad \left. + \exp\{-C\varphi\} (\Delta\varphi)_{k\bar{l}} \right) \end{aligned}$$

$$\begin{aligned}
 &= C^2 \exp\{-C\varphi\} \left(\sum_{k\bar{l}} g^{k\bar{l}} \varphi_k \varphi_{\bar{l}} \right) (m + \Delta\varphi) \\
 &\quad - C \exp\{-C\varphi\} \left(\sum_{k,l} g^{k\bar{l}} \varphi_{\bar{l}} (\Delta\varphi)_k \right) \\
 &\quad - C \exp\{-C\varphi\} \left(\sum_{k,l} g^{k\bar{l}} \varphi_k (\Delta\varphi)_{\bar{l}} \right) \\
 (2.18) \quad &\quad - C \exp\{-C\varphi\} (\Delta'\varphi)(m + \Delta\varphi) + \exp\{-C\varphi\} \Delta'(\Delta\varphi)
 \end{aligned}$$

Recall the Schwarz' inequality. In \mathbb{C} , if u, v are two complex numbers, and the inner product on \mathbb{C} given by $\langle y, v \rangle = u\bar{v}$, then $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\operatorname{Re}u\bar{v}$.

Looking at just the first three terms, we have

$$\begin{aligned}
 &\exp\{-C\varphi\} \frac{1}{(m + \Delta\varphi)} \left(\langle C(m + \Delta\varphi)\nabla\varphi, C(m + \Delta\varphi)\nabla\varphi \rangle_{g_{i\bar{j}}} \right. \\
 &\quad \left. - \langle C(m + \Delta\varphi)\nabla\varphi, \nabla(\Delta\varphi) \rangle_{g_{i\bar{j}}} + \langle \nabla\Delta\varphi, C(m + \Delta\varphi)\nabla\varphi \rangle_{g_{i\bar{j}}} \right)
 \end{aligned}$$

Thus, since $\|u\|^2 + \|v\|^2 - 2\operatorname{Re}u\bar{v} = \|u - v\|^2 \geq 0$, and substituting back into (2.18),

$$\begin{aligned}
 \Delta'(\exp\{-C\varphi\}(m + \Delta\varphi)) &\geq -(m + \Delta\varphi)^{-1} \exp\{-C\varphi\} \sum_{i,j} g^{i\bar{j}} (\Delta\varphi)_i (\Delta\varphi)_{\bar{j}} \\
 &\quad - C \exp\{-C\varphi\} (\Delta'\varphi)(m + \Delta\varphi) + \exp\{-C\varphi\} \Delta'(\Delta\varphi)
 \end{aligned}$$

Now, since we want to uniformly bound $(m + \Delta\varphi)$, and we have $\Delta'(\Delta\varphi)$, we consider,

$$\begin{aligned}
 &-(m + \Delta\varphi)^{-1} \sum_{i,j} g^{i\bar{j}} (\Delta\varphi)_i (\Delta\varphi)_{\bar{j}} + \Delta'(\Delta\varphi) \\
 &\geq -(m + \Delta\varphi)^{-1} \sum_{i,j} g^{i\bar{j}} (\Delta\varphi)_i (\Delta\varphi)_{\bar{j}} \\
 &\quad + \Delta F + \sum_{i,j,k} \frac{1}{(1 + \varphi_{i\bar{i}})} \frac{1}{(1 + \varphi_{k\bar{k}})} \|\varphi_{i\bar{k}j}\|^2 \\
 &\quad + \sum_{i,l} \left(\inf_{i \neq l} R_{i\bar{i}l\bar{l}} \right) \left[\sum_{i,l} \frac{1 + \varphi_{l\bar{l}}}{1 + \varphi_{i\bar{i}}} - m^2 \right]
 \end{aligned}$$

But,

$$\begin{aligned}
& (m + \Delta\varphi)^{-1} \sum_i \frac{1}{(1 + \varphi_{1\bar{i}})} (\Delta\varphi)_i (\Delta\varphi)_{\bar{i}} \\
&= (m + \Delta\varphi)^{-1} \left(\sum_i \frac{1}{(1 + \varphi_{i\bar{i}})} (|(\Delta\varphi)_{\bar{i}}|^2) \right) \\
&= (m + \Delta\varphi)^{-1} \left(\sum_i \frac{1}{(1 + \varphi_{i\bar{i}})} \left(\left| \sum_k \varphi_{k\bar{k}i} \right|^2 \right) \right) \\
&= (m + \Delta\varphi)^{-1} \left(\sum_i \frac{1}{(1 + \varphi_{i\bar{i}})} \left(\left| \sum_k \frac{\varphi_{k\bar{k}i}}{(1 + \varphi_{k\bar{k}})^{1/2}} (1 + \varphi_{k\bar{k}})^{1/2} \right|^2 \right) \right) \\
&\geq (m + \Delta\varphi)^{-1} \left(\sum_i \frac{1}{(1 + \varphi_{i\bar{i}})} \left(\sum_k \frac{|\varphi_{k\bar{k}i}|^2}{1 + \varphi_{k\bar{k}}} \right) \left(\sum_k (1 + \varphi_{k\bar{k}}) \right) \right) \\
&\geq (m + \Delta\varphi)^{-1} \underbrace{\left(\sum_k (1 + \varphi_{k\bar{k}}) \right)}_{(m + \Delta\varphi)} \left(\sum_{i,k} \frac{|\varphi_{k\bar{k}i}|^2}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} \right) \\
&= \sum_{i,k} \frac{|\varphi_{k\bar{k}i}|^2}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})} \\
(2.19) \quad &\geq \sum_{i,j,k} \frac{1}{(1 + \varphi_{i\bar{i}})(1 + \varphi_{k\bar{k}})}
\end{aligned}$$

So, we have

$$\begin{aligned}
& \Delta'(\exp\{-C\varphi\})(m + \Delta\varphi) \\
&\geq \exp\{-C\varphi\} \left(\Delta\varphi + \left(\inf_{i \neq l} R_{i\bar{i}l\bar{l}} \left[\sum_{i,l} \frac{1 + \varphi_{l\bar{l}}}{1 + \varphi_{i\bar{i}}} - m^2 \right] \right) \right) \\
(2.20) \quad & -C \exp\{-C\varphi\} (\Delta'\varphi)(m + \Delta\varphi)
\end{aligned}$$

Note the following identity,

$$\begin{aligned}
(2.21) \quad \Delta'\varphi &= \sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \\
&= m - \sum_i \frac{1}{1 + \varphi_{i\bar{i}}}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
 & \Delta'(\exp\{-C\varphi\}(m + \Delta\varphi)) \\
 & \geq \exp\{-C\varphi\} \left(\Delta F - m^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}} \right) \\
 & \quad + \exp\{-C\varphi\} \inf_{i \neq l} R_{i\bar{i}l\bar{l}} \left(\sum_{i,l} \frac{1 + \varphi_{l\bar{l}}}{1 + \varphi_{i\bar{i}}} \right) \\
 & \quad - C \exp\{-C\varphi\}(m + \Delta\varphi) \left(m - \sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \right) \\
 & = \exp\{-C\varphi\} (\Delta F - m^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}}) \\
 & \quad + \exp\{-C\varphi\} \inf_{i \neq l} R_{i\bar{i}l\bar{l}} \left(\sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \right) (m + \Delta\varphi) \\
 (2.22) \quad & \quad + C \exp\{-C\varphi\}(m + \Delta\varphi) \left(\sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \right) \\
 & \quad - C \exp\{-C\varphi\} m(m + \Delta\varphi) \\
 & = \exp\{-C\varphi\} \left(\Delta F - m^2 \inf_{i \neq l} R_{i\bar{i}l\bar{l}} \right) \\
 (2.23) \quad & \quad - C \exp\{-C\varphi\} m(m + \Delta\varphi) \\
 & \quad + \left(C + \inf_{i \neq l} R_{i\bar{i}l\bar{l}} \right) \exp\{-C\varphi\}(m + \Delta\varphi) \left(\sum_i \frac{1}{1 + \varphi_{i\bar{i}}} \right)
 \end{aligned}$$

3. S IS OPEN.

In this section, we follow Dominic Joyce's book, Compact Manifolds with Special Holonomy. The following theorem shows that when φ is a solution, then we can find solutions in sufficiently small neighborhoods.

Theorem 3.1. *Let (M, g) be a compact complex manifold, and g a Kähler metric on M , with Kähler form ω . Fix $\alpha \in (0, 1)$, and suppose that $f' \in C^{3,\alpha}(M)$, $\varphi \in C^{5,\alpha}(M)$, and $A > 0$ satisfy the equations*

$$(3.1) \quad \int_M \varphi dVol_g = 0 \text{ and } (\omega + \partial\bar{\partial}\varphi)^m = A \exp\{f\} \omega^m$$

Then, whenever $f' \in C^{3,\alpha}(M)$, and $\|f' - f\|_C^{3,\alpha}$ is sufficiently small, there exists $\varphi' \in C^{5,\alpha}(M)$ and $A' > 0$ such that

$$(3.2) \quad \int_M \varphi' dVol_g = 0 \text{ and } (\omega + \partial\bar{\partial}\varphi')^m = A' \exp\{f'\}\omega^m.$$

Proof. Define X to be the vector subspace of $\varphi \in C^{5,\alpha}(M)$ for which $\int_M \varphi dVol_g = 0$. Let U be the subset of X such that $\omega + \partial\bar{\partial}\varphi$ is a positive $(1,1)$ -form on M . Then it's not hard to see that U is open in X . Let Y be the Banach space $C^{3,\alpha}(M)$. Suppose that $\varphi \in U$ and $a \in \mathbb{R}$.

□

APPENDIX A

In this appendix, we verify the formula for the derivative of the inverse matrix of the metric.

APPENDIX B

Here we verify the formula for the derivative of the logarithm of a determinant with respect to the variable z^k .

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, 480 LINCOLN DRIVE,
MADISON, WI 53706

E-mail address: `lien@math.wisc.edu`