NOTES ON SCHAUDER THEORY: INTERIOR ESTIMATES

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Abstract. The calculations follow Gilbarg and Trudinger Chapter 4 and 6. These are written for my own enlightenment, and I've included some trivial details that they've rightly omitted.

1. Some notation and some estimates we take for granted

1.1. Notation.

$D_i u = \dfrac{\partial u}{\partial x^i}$,
$D_{ij} u = \dfrac{\partial^2 u}{\partial x^i \partial x^j}$,
$Du = (D_1 u, D_2 u, \ldots, D_n u) =$ gradient of u,
$\omega_n =$ volume of unit ball in $\mathbb{R}^n$,
$\nu =$unit outward normal,
$ds =$ surface area element, codimension 1.
We assume throughout that $n > 2$.

1.2. **The estimates we take for granted.** We assume that f is locally Hölder continuous, and bounded. We denote by $w(x) = \int_{B_2} \Gamma(x-y) f(y) dy$, the Newtonian Potential of f, where Γ is the fundamental solution to Laplace's equation. ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

$$D_{ij} w(x) = \int_{B_2} D_{ij}\Gamma(x-y)(f(y) - f(x))dy - f(x)\int_{\partial B_2} D_i \Gamma(x-y)\nu_j(y) ds_y, \tag{1.1}$$

$$|D_i \Gamma(x-y)| \leq \frac{1}{n\omega_n}|x-y|^{1-n}$$
$$|D_{ij} \Gamma(x-y)| \leq \frac{1}{\omega_n}|x-y|^{-n} \tag{1.2}$$

2. The Hölder estimates of the second derivatives

Lemma 2.1. *Let B_1 and B_2 be concentric balls in $\mathbb{R}^n$ around x_0 or radius R. Let $f \in C^{\alpha}(\bar{B}_2)$ for $0 < \alpha < 1$. Then*

$$w(x) = \int_{B_2} \Gamma(x-y) f(y) dy,$$

, the Newtonian Potential of f in B_2, is in $C^{2,\alpha}(B_2)$ and

$$|D^2 w|_{0;B_1} + R^{\alpha}[D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^{\alpha}[f]_{\alpha;B_2}), \tag{2.1}$$

where C depends on n and alpha, $|f|_0$ denotes the sup norm, and $[f]_{\alpha;B_2}$ denotes the Hölder norm over the domain B_2.

Proof. For any $x \in B_1$, we have by (1.1)

$$D_{ij}w(x) = \int_{B_2} D_{ij}\Gamma(x-y)(f(y)-f(x))dy - f(x)\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)ds_y,$$

so by (1.2), we have

$$\begin{aligned} |D_{ij}w(x)| &\leq \frac{|f(x)|}{n\omega_n}R^{1-n}\int_{\partial B_2} ds_y + \frac{[f]_{\alpha;x}}{\omega_n}\int_{B_2}|x-y|^{\alpha-n}dy \\ &\leq 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{\alpha;x} \\ &\leq C_1(|f(x)| + R^\alpha [f]_{\alpha;x}, \end{aligned} \tag{2.2}$$

where $C_1 = C_1(n,\alpha)$. Similarly, for any other point $\bar{x} \in B_1$, we have by (1.1)

$$D_{ij}w(\bar{x}) = \int_{B_2} D_{ij}\Gamma(\bar{x}-y)(f(y)-f(\bar{x}))dy - f(\bar{x})\int_{\partial B_2} D_i\Gamma(\bar{x}-y)\nu_j(y)ds_y.$$

Let $\delta = |x-\bar{x}|, \xi = \frac{1}{2}(x+\bar{x})$, and $B_\delta = B_\delta(\xi)$ then by subtraction, we have

$$\begin{aligned} &D_{ij}w(\bar{x}) - D_{ij}w(x) = \\ &\quad = (f(x)-f(\bar{x}))\int_{\partial B_2} D_{ij}\Gamma(\bar{x}-y)\nu_j(y)ds_y && (I_1)\\ &\quad + f(x)\int_{\partial B_2}(D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)\nu_j(y)ds_y && (I_2)\\ &\quad + \int_{B_\delta} D_{ij}\Gamma(x-y)(f(x)-f(y))dy && (I_3)\\ &\quad + \int_{B_\delta} D_{ij}\Gamma(\bar{x}-y)(f(y)-f(\bar{x}))dy && (I_4)\\ &\quad + (f(x)-f(\bar{x}))\int_{B_2-B_\delta} D_{ij}\Gamma(x-y)dy && (I_5)\\ &\quad + \int_{B_2-B_\delta}(D_{ij}\Gamma(x-y) - D_{ij}\Gamma(\bar{x}-y))(f(\bar{x})-f(y))dy && (I_6) \end{aligned}$$

We want to estimate each of these terms by something not dependent upon w. We start with I_2.

$$
\begin{aligned}
I_2 &= f(x) \int_{\partial B_2} D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)\nu_j(y)ds_y \\
|I_2| &\le |f|_{0;x} \int_{\partial B_2} |D_i\Gamma(x-y) - D_i\Gamma(\bar{x}-y)|ds_y \\
&\le |f|_{0;x}\, |x-\bar{x}| \int_{\partial B_2} |DD_i\Gamma(\tilde{x}-y)|ds_y \qquad \text{for some } \tilde{x} \text{ between } x \text{ and } \bar{x} \\
&\le |f|_{0;x}\, |x-\bar{x}| \int_{\partial B_2} \frac{n}{\omega_n}|\tilde{x}-y|^{-n}ds_y \\
&\qquad \left(\text{use } |DD_i\Gamma(\tilde{x}-y)| \le \sum_j |D_{ij}\Gamma(\tilde{x}-y)| \le \frac{n}{\omega_n}|\tilde{x}-y|^{-n}\right) \\
&\le |f|_{0;x}\, |x-\bar{x}|\, \frac{n}{\omega_n} R^{-n} \int_{\partial B_2} ds_y, \qquad \text{since } |\tilde{x}-y| \le R \text{ for } y \in \partial B_2 \\
&\le |f|_{0;x}\, |x-\bar{x}|\, \frac{n}{\omega_n} R^{-n} \left(n\omega_n (2R)^{n-1}\right) \\
&= |f|_{0;x}\, |x-\bar{x}|\, \frac{n^2 2^{n-1}}{R} \\
&= |f|_{0;x}\, n^2 2^{n-1} \left(\frac{\delta}{R}\right) \\
&\le |f|_{0;x}\, n^2 2^{n-\alpha} \left(\frac{\delta}{R}\right)^{\alpha}
\end{aligned}
$$

And now, I_1.

$$
\begin{aligned}
I_1 &= (f(x) - f(\bar{x})) \int_{\partial B_2} D_i\Gamma(\bar{x}-y)\nu_j(y)ds_y \\
|I_1| &\le [f]_{\alpha;x}\, \delta^\alpha \int_{\partial B_2} |D_i\Gamma(\bar{x}-y)|ds_y \\
&\le [f]_{\alpha;x}\, \delta^\alpha \int_{\partial B_2} \frac{1}{n\omega_n}|x-y|^{1-n}ds_y \\
&\le [f]_{\alpha;x}\, \delta^\alpha \int_{\partial B_2} \frac{1}{n\omega_n}R^{1-n}ds_y \\
&= [f]_{\alpha;x}\, \delta^\alpha \frac{1}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_y \\
&= [f]_{\alpha;x}\, \delta^\alpha 2^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_{B_\delta} D_{ij}\Gamma(x-y)(f(x)-f(y))dy \\
|I_3| &\le \int_{B_{3\delta/2(x)}} |D_{ij}\Gamma(x-y)||f(x)-f(y)|dy \\
&\le \int_{B_{3\delta/2(x)}} \frac{1}{\omega_n}|x-y|^{\alpha-n}\frac{|f(x)-f(y)|}{|x-y|^\alpha}dy \\
&\le \frac{1}{\omega_n}[f]_{\alpha;x}\int_{B_{3\delta/2(x)}} |x-y|^{\alpha-n}\,dy \\
&= \frac{n}{\alpha}\left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;x}
\end{aligned}
$$

The estimation of I_4 is identical, and we have:

$$
\begin{aligned}
|I_4| &= \int_{B_\delta} |D_{ij}\Gamma(\bar{x}-y)||f(y)-f(\bar{x})|\,dy \\
&\le \frac{n}{\alpha}\left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;\bar{x}}
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= (f(x)-f(\bar{x}))\int_{B_2-B_\delta} D_{ij}\Gamma(x-y)dy \\
|I_5| &= |f(x)-f(\bar{x})|\left|\int_{B_2-B_\delta} D_{ij}\Gamma(x-y)dy\right| \\
&\le [f]_\alpha|x-\bar{x}|^\alpha\left|\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)\,ds_y - \int_{\partial B_\delta(\xi)} D_i\Gamma(x-y)\nu_j(y)\,ds_y\right| \\
&\le [f]_\alpha\delta^\alpha\left|\int_{\partial B_2} D_{ij}\Gamma(x-y)\nu_j(y)\,ds_y\right| + \left|\int_{\partial B_\delta(\xi)} D_i\Gamma(x-y)\nu_j(y)\,ds_y\right| \\
&\le [f]_\alpha\delta^\alpha\left(2^{n-1} + \frac{1}{n\omega_n}\int_{\partial B_\delta(\xi)} \frac{1}{|x-y|^{1-n}}\,ds_y\right) \\
&\le [f]_\alpha\delta^\alpha\left(2^{n-1} + \frac{1}{n\omega_n}\left(\frac{\delta}{2}\right)^{1-n}\int_{\partial B_\delta(\xi)} ds_y\right) \\
&\le [f]_\alpha\delta^\alpha\left(2^{n-1} + \frac{1}{n\omega_n}\left(\frac{\delta}{2}\right)^{1-n}(\delta)^{n-1}\,n\omega_n\right) \\
&= [f]_\alpha\delta^\alpha 2^n
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= \int_{B_2 - B_\delta} (D_{ij}\Gamma(x-y) - D_{ij}\Gamma(\bar{x}-y))(f(\bar{x}) - f(y))\, dy \\
|I_6| &\le |x-\bar{x}| \int_{B_2 - B_\delta} |DD_{ij}(\tilde{x}-y)||f(\bar{x}) - f(y)|\, dy \qquad \text{for } \tilde{x} \text{ between } x \text{ and } \bar{x} \\
&\le \frac{n(n+5)}{\omega_n}\, \delta \int_{B_2 - B_\delta} \frac{|f(\bar{x}) - f(y)|}{|\tilde{x}-y|^{n+1}}\, dy \\
&\le \frac{n(n+5)}{\omega_n}\, \delta\, [f]_\alpha \int_{B_2 - B_\delta} \frac{|\bar{x}-y|}{|\tilde{x}-y|^{n+1}}\, dy \\
&\le \frac{n(n+5)}{\omega_n}\, \delta\, [f]_\alpha \int_{|y-\xi|\ge\delta} \frac{\left(\frac{3}{2}|\xi-y|\right)^\alpha}{\left(\frac{1}{2}|\xi-y|\right)^{n+1}}\, dy \\
&\qquad\qquad\qquad\qquad \text{since } |\bar{x}-y| \le \frac{3}{2}|\xi-y| \le 3|\tilde{x}-y| \\
&\le \frac{n(n+5)}{\omega_n}\, \delta\, [f]_\alpha \left(\frac{3}{2}\right)^\alpha 2^{n+1} \int_{|y-\xi|\ge\delta} \frac{1}{|\xi-y|^{n+1-\alpha}}\, dy \\
&\le \frac{n(n+5)}{\omega_n}\, \delta\, [f]_\alpha \left(\frac{3}{2}\right)^\alpha 2^{n+1} \int_\delta^{3R} n\omega_n \frac{1}{r^{n+1-\alpha}} r^{n-1}\, dr \\
&\le n^2(n+5)\, \delta\, [f]_\alpha \left(\frac{3}{2}\right)^\alpha 2^{n+1} \frac{1}{\alpha-1} \left((3R)^{\alpha-1} - \delta^{\alpha-1}\right)
\end{aligned}
$$

So,

$$
|I_6| \le \frac{n^2(n+5)}{1-\alpha}\, \delta^\alpha [f]_\alpha \left(\frac{3}{2}\right)^\alpha 2^{n+1}
$$

Adding up all six of these terms, we have

$$
|D_{ij}w(\bar{x}) - D_{ij}w(x)| \le C(n,\alpha)\left(R^{-\alpha}\delta^\alpha |f|_{0;B_2} + \delta^\alpha [f]_\alpha\right) \tag{2.3}
$$

C only depends on n and α. Thus, multiplying through by $\delta^{-\alpha}$ and R^α, we have

$$
R^\alpha \frac{|D_{ij}w(\bar{x}) - D_{ij}w(x)|}{|x-\bar{x}|} \le C(n,\alpha)\left(|f|_{0;B_2} + R^\alpha [f]_\alpha\right).
$$

Combining this with the previous estimate, we have (2.1). This only bounds the sup norm of the second derivatives and the Hölder semi-norm. However, $|Dw|_0$ and $|w|_0$ follow easily by the same type of computations as above, and the identities presented in the last section. In fact, we have that

$$
\begin{aligned}
\|D_i w(x)\|_{0;B_1} &= \int_{B_1} |D_i\Gamma(x-y)f(y)|\,dy \\
&\le \|f\|_{0;B_1} \int_{B_1} |D_i\Gamma(x-y)|\,dy \\
&\le \|f\|_{0;B_1} \int_{B_1} \frac{1}{|x-y|^{n-1}}\,dy \\
&\le \|f\|_{0;B_1} \frac{1}{\omega_n} \int_{0\le r\le R} 1\,dr \\
&\le \|f\|_{0;B_1} R
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|w(x)|_{0;B_1} &= \int_{B_1} \Gamma(x-y)f(y)\,dy \\
&\le |f|_{0;B_1} \int_{B_1} \frac{1}{|x-y|^{n-2}}\,dy \\
&\le |f|_{0;B_1} \int_{|z|\le R} \frac{1}{|z|^{n-2}}\,dz \\
&\le |f|_{0;B_1} \frac{1}{n\omega_n} \int_{r\le R} \frac{1}{r^{n-2}} r^{n-1}\,dr \\
&\le |f|_{0;B_1} \frac{R^2}{2}
\end{aligned}
$$

□

Remark. Throughout, we have that in bounding the various semi-norms, our constants often depends on R, or the size of the ball that we had chosen at the beginning. Now, since our domain is bounded, we want to exploit the fact that there is an upper bound on R, though not necessarily a lower bound. Hence, in (2.3), we clear the denominators of R. This is where Gilbarg and Trudinger begins to define weighted norms. This is made more apparent in the subsequent discussion.

With the bounds on the Newtonian Potential, by Green's representation theorem, if u is a solution to Poisson's equation $\Delta u = f$ in a ball B_{2R}, and $f \in C^\alpha(B_{2R})$, then we have

$$u = w + h,$$

where w is the Newtonian Potential of f, and h is some harmonic function on B_{2R}.
Thus, we have

$$\|u\|_{2,\alpha;B_1} = \|u\|_{0,B_1} + \|Du\|_{0;B_1} + \|D^2u\|_{0,B_1} + [D^2u]_{0,\alpha;B_1} \tag{2.4}$$

We bound each of these separately, and also reformulate these in terms of a weighted norm, when we wish to clear the denominators.

$$
\begin{aligned}
|Du|_{0;B_1} &= |Dw + Dh|_{0;B_1} \\
&\le |Dw|_{0;B_1} + |Dh|_{0;B_1} \\
&\le R|f|_{0;B_1} + \frac{n}{R}|h|_{0,B_2}
\end{aligned}
$$

(see last section for bounds on derivatives of harmonic functions)

(If we clear denominators, we have

$$R|Du|_{0;B_1} \le R^2|f|_{0;B_1} + n|h|_{0,B_2}. \tag{2.5}$$

)

$$
\begin{aligned}
|D^2u|_{0;B_1} &= |D^2w + D^2h|_{0;B_1} \\
&\le |D^2w|_{0;B_1} + |D^2h|_{0;B_1} \\
&\le (2^{n-1}|f|_{0;B_2} + \frac{n}{\alpha}(3R)^\alpha[f]_{\alpha;B_2}) + (\frac{2n}{R})^2|h|_{0;B_2}
\end{aligned}
$$

(Clearing denominators, we have the following inequality:

$$|D^2u|_{0;B_1} \le (2n)^2|h|_{0;B_2} + 2^{n-1}R^2|f|_{0;B_2} + \frac{n3^\alpha}{\alpha}R^{2+\alpha}[f]_{\alpha;B_2} \tag{2.6}$$

)

$$
\begin{aligned}
[D^2u]_{\alpha;B_1} &\le [D^2u]_{\alpha;B_1} + [D^2h]_{\alpha;B_1} \\
&\le C(n,\alpha)(R^{-alpha}|f|_{0:B_2} + [f]_{\alpha;B_2}) + [D^2h]_{\alpha;B_1}.
\end{aligned}
$$

We have

$$
\begin{aligned}
[D^2h]_{\alpha;B_1} &= \sup_{x,y\in B_1} \frac{|D^2h(x) - D^2h(y)|}{|x-y|^\alpha} \\
&\le \sup_{x,y\in B_1} R^{1-\alpha}\frac{|D^2h(x) - D^2h(y)|}{|x-y|} \\
&\le 2R^{1-\alpha} \sup_{x,y\in B_1} |D^3h| \\
&\le (2R)^{1-\alpha}\left(\frac{3n}{R}\right)^3 |h|_{0;B_2} \\
&\le \frac{C}{R^{2+\alpha}}|h|_{0;B_2}
\end{aligned}
$$

Thus, this gives us

$$[D^2u]_{\alpha;B_1} \le C(n,\alpha)\left(R^{-alpha}|f|_{0:B_2} + [f]_{\alpha;B_2} + R^{-2-\alpha}|h|_{0;B_2}\right)$$

(Reformulated, so that we can be arbitrarily close to the boundary, we have

$$R^{2+\alpha}[D^2u]_{\alpha;B_1} \le C(n,\alpha)\left(R^2|f|_{0:B_2} + R^{2+\alpha}[f]_{\alpha;B_2} + |h|_{0;B_2}\right) \tag{2.7}$$

)

To take care of $|h|_{0;B_2}$, we recognize that $h = u - w$, so $|h|_{0;B_2} \leq |u|_{0;B_2} + |w|_{0;B_2}$, and the supremum of the Newtonian potential of f depends on the $|f|_{0;B_2}$ by Green's representation theorem.

Adding all these together, we have the following lemma with the weighted norms.

Lemma 2.2. *If $u \in C^2(B_{2R})$ is a solution to Poisson's equation, i.e. $\Delta u = f$, for $f \in C^\alpha B_{2R}$, then $u \in C^{2,\alpha}(B_{2R})$, and we have*

$$\|u\|'_{2,\alpha;B_1} \leq C(n,\alpha)(\|u\|_{0,B_2} + R^2\|f\|'_{0,\alpha}). \tag{2.8}$$

Now, this inequality is specific to a ball of fixed radius. For a general bounded domain Ω, we need to alter our norms slightly by $d_x = \operatorname{dist}(x, \partial\Omega)$.

For fixed $x \in \Omega$, let $R = \frac{1}{3}d_x$. Then,

$$\begin{aligned} d_x|Du(x)| + d_x^2|D^2u(x)| &\leq (3R)|Du|_{0;B_1} + (3R)^2|D^2u|_{0;B_1} \\ &\leq C(\|u\|_{0;B_2} + R^2\|f\|'_{0,\alpha;B_2}) \qquad \text{by (2.8)} \\ &\leq C(\|u\|_{0;B_2} + |f|^{(2)}_{0,\alpha;B_2}), \end{aligned}$$

and we have

$$|u|^*_{2;\Omega} \leq C(\|u\|_{0,B_2} + \|f\|^{(2)}_{0,\alpha;\Omega}), \tag{2.9}$$

where $|u|^*_{k,\alpha;\Omega}$ is a norm weighted with distance to the boundary, rather than a fixed R, and $|f|^{(2)}_{0,\alpha;B_2}$ is the same but with an extra factor of d_x^2.

3. Something other than the Laplacian

3.1. **Constant coefficient.** We first consider an elliptic second order differential operator with constant coefficients. Let $L_0 = a^{ij}D_{ij}$, where a^{ij} are the entries of a constant matrix A. We say that L_0 is elliptic if there exists a Λ so that

$$\frac{1}{\Lambda}|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for } \xi \in \mathbb{R}^n$$

Note that for the Laplacian, $A = Id$. Suppose that $L_0u = f \in C^\alpha(\Omega)$. We wish to establish the same identity

$$\|u\|^*_{2,\alpha;\Omega} \leq C\left(\|u\|_{0;\Omega} + \|f\|^{(2)}_{0,\alpha;\Omega}\right), \tag{3.1}$$

where $C = C(n,\alpha)$.

Proof. Let P be a nonsingular linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $y = Px$. Then this transformation takes $u(x)$ to $\tilde{u}(y) : P(\Omega) \to \mathbb{R}$, i.e. for $y \in P(\Omega), \tilde{u}(y) = u(x)$.

In fact, this transformation takes $a^{ij}D_{ij}u(x) = f(x)$ to the equation $\tilde{a}^{ij}D_{ij}\tilde{u}(y) = \tilde{f}(y)$, where $\tilde{a}^{ij}$ are coefficients of the matrix PAP^t, with $P^t = P$ transpose. We try and see this a little better.

$$
\begin{aligned}
u(x) &\xrightarrow{\quad P \quad} \tilde{u}(y)\\
D_i u(x) &\xrightarrow{\quad P \quad} D_i(\tilde{u}(Px))\\
&\qquad = \sum_k \frac{\partial \tilde{u}}{\partial x^k}(Px)\, p_{ki}\\
D_{ij} u(x) &\xrightarrow{\quad P \quad} D_j\left(\sum_k \frac{\partial \tilde{u}}{\partial x^k}(Px) p_{ki}\right)\\
&\qquad = \sum_k D_j\left(\frac{\partial \tilde{u}}{\partial x^k}(Px)\right) p_{ki}\\
&\qquad = \sum_k \left(\sum_l \frac{\partial^2 \tilde{u}}{\partial x^l \partial x^k}(Px)\, p_{ki}\right) p_{lj}\\
&\qquad = \sum_{k,l} p_{ki}\, p_{lj}\, \frac{\partial^2 \tilde{u}}{\partial x^l \partial x^k}(Px)\\
a^{ij} D_{ij} u(x) &\xrightarrow{\quad P \quad} \sum_{i,j}\sum_{k,l} p_{ki} a^{ij} p_{lj}\, \frac{\partial^2 \tilde{u}}{\partial x^l \partial x^k}(Px)\\
&\qquad = \sum_{k,l}\left(\sum_{i,j} p_{ki} a^{ij} p_{lj}\right) \frac{\partial^2 \tilde{u}}{\partial x^l \partial x^k}(Px)\\
&\qquad = \sum_{k,l} \tilde{a}^{kl}\, \frac{\partial^2 \tilde{u}}{\partial x^l \partial x^k}(Px)\\
&\qquad = \tilde{f}(Px)
\end{aligned}
$$

Since A is a symmetric real matrix, there is an orthognal P so that that $\tilde{A}$ is a diagonal matrix with the diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. Let D be the diagonal matrix with the i^{th} diagonal element $\frac{1}{\sqrt{\lambda_i}}$.

Let $Q = DP$, and $y = Qx$. Under this transformation, $\tilde{A} = DPAP^tD^t = Id$.

Hence, $L_0 u(x) = f(x) \xrightarrow{Q} \Delta \tilde{u}(y) = \tilde{f}(y)$. We assume furthermore that Q takes the upper half space, $x_n > 0$ to the upper half space $y_n > 0$.

Since P preserves length, we have

$$
\frac{1}{\sqrt{\Lambda}}|x| \leq |Qx| \leq \sqrt{\Lambda}|x|
$$

So, there exists a c so that

$$
\begin{aligned}
c^{-1}\|u\|^*_{k,\alpha;\Omega} &\leq \|\tilde{u}\|^*_{k,\alpha;\tilde{\Omega}} \leq c\|u\|^*_{k,\alpha;\Omega}\\
c^{-1}\|u\|^{(k)}_{0,\alpha;\Omega} &\leq \|\tilde{u}\|^{(k)}_{0,\alpha;\tilde{\Omega}} \leq c\|u\|^{(k)}_{0,\alpha;\Omega}
\end{aligned}
$$

Thus,

$$\begin{aligned}\|u\|^*_{2,\alpha;\Omega} &\le c\|\tilde{u}\|^*_{2,\alpha;\tilde{\Omega}} \\ &\le C(\|\tilde{u}\|_{0,\tilde{\Omega}} + \|\tilde{f}\|^{(2)}_{0,\alpha;\tilde{\Omega}} \\ &\qquad\qquad \text{by our previous result, (2.9)} \\ &\le C(\|u\|_0 + \|f\|^{(2)}_{0,\alpha;\Omega})\end{aligned}$$

□

3.2. **Hölder continuous coefficients.** Now, we wish to extend the result (3.1) to elliptic operators with Hölder continuous coefficients. Let $L = a^{ij}D_ij + b^iD_i + c$ be an elliptic operator in a domain Ω. That is,

$$a^{ij}\xi_i\xi_j \ge \frac{1}{\Lambda}\|\xi\|^2 \qquad \text{for every } x \in \Omega, \xi \in \mathbb{R}^n \tag{3.2}$$

and

$$\|a^{ij}\|^{(0)}_{0,\alpha;\Omega}, \|b^i\|^{(1)}_{0,\alpha;\Omega}, \|c\|^{(2)}_{0,\alpha;\Omega} \le \Lambda. \tag{3.3}$$

Then if $u \in C^{2,\alpha}(\Omega)$ is a solution to Poisson's equation, $Lu = f$, where $f \in C^\alpha(\Omega)$, we have that

$$|u|^*_{2,\alpha;\Omega} \le C\left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}\right)$$

where C depends on n, α, Λ.

Proof. Let $x_0 \in \Omega$. Define the constant coefficient elliptic operator $L_0 = a^{ij}(x_0)D_{ij}$. Thus, since $Lu = f$,

$$L_0u = (a^{ij}(x_0) - a^{ij}(x))D_{ij}u - b^iD_iu - cu + f,$$

in Ω.

Denote the right hand side by F.

Remark. Here, we have a constant coefficient operator, and so we can apply our previous result and obtain that

$$\|u\|^*_{2,\alpha;\Omega} \le C(\|u\|_{0,\Omega} + \|F\|^{(2)}_{0,\alpha;\Omega}$$

Now, in bounding F, however, we get the seminorms of $\|u\|^*_{2,\alpha;\Omega}$ showing up on the right hand side. Thus, we would like to bound $\|F\|^{(2)}_{0,\alpha;\Omega}$ so that the seminorm terms are dependent upon a constant μ of our choice. We do this by choosing a ball of a particular size in a neighborhood of x_0.

Let $\mu \le \frac{1}{2}$ be a positive constant to be determined later. Denote $d = \mu d_{x_0}$, and $B = B_d(x_0)$. Then by our previous result, (3.1), we have for our ball, B,

$$\|u\|^*_{2,\alpha;B} \le C\left(\|u\|_{0;B} + \|F\|^{(2)}_{0,\alpha;B}\right) \tag{3.4}$$

It's clear that $\|u\|_{0;B} \le \|u\|_{0,\Omega}$. In general, for functions u, v in a "nice" domain, A, we have

$$\|uv\|^{(\sigma)}_{k,\alpha;A} \le \|u\|^{(\rho)}_{k,\alpha;A}\|v\|^{(\tau)}_{k,\alpha;A},$$

where $\rho+\tau=\sigma$. We show this only for the case that $k=0, \sigma=2, \rho=2, \tau=0$.

$$\begin{aligned}
\|uv\|^{(2)}_{0,\alpha;A} &= \sup_{x\in A} d_x^2|uv| + \sup_{x,y\in A} d_{x,y}^{2+\alpha}\frac{|uv(x)-uv(y)|}{|x-y|^\alpha} \\
&\le \sup_{x\in A} d_x^2|u||v| \\
&\qquad + \sup_{x,y\in A} d_{x,y}^{2+\alpha}\frac{|u(x)v(x)-u(x)v(y)+u(x)v(y)-u(y)v(y)|}{|x-y|^\alpha} \\
&\le \left(\sup_{x\in A} d_x^2|u|\right)\sup_{x\in A}|v| \\
&\qquad + \sup_{x,y\in A} d_{x,y}^{2+\alpha}\left(\frac{|u(x)v(x)-u(x)v(y)|}{|x-y|^\alpha}+\frac{|u(x)v(y)-u(y)v(y)|}{|x-y|^\alpha}\right) \\
&\le |u|^{(2)}_{0;A}|v|_{0;A} + |u|^{(2)}_{\alpha;A}|v|^{(0)}_{\alpha;A} + |u|^{(2)}_{0,\alpha;A}|v|_{0;A} \\
&\le |v|_{0,A}\left(\|u\|^{(2)}_{0,\alpha;A}\right) + \|u\|^{(2)}_{0,\alpha;A}|v|^{(0)}_{\alpha;A} \\
&\le \|u\|^{(2)}_{0,\alpha;A}\|v\|^{(0)}_{0,\alpha;A}
\end{aligned}$$

Thus, we have,

$$\begin{aligned}
\|F\|^{(2)}_{0,\alpha;B} \le \|(a^{ij}(x_0)-a^{ij}(x))\|^{(0)}_{0,\alpha;B}\|D_{ij}u\|^{(2)}_{0,\alpha;B} + \|b_i\|^{(1)}_{0,\alpha;B}\|D_i\|^{(1)}_{0,\alpha;B} \\
+ \|c\|^{(2)}_{0,\alpha;B}\|u\|^{(0)}_{0,\alpha;B} + \|f\|^{(2)}_{0,\alpha;B}
\end{aligned}$$

We look at each of these terms individually, focusing first on the derivatives of u. Note that for $x\in B$, $d=\mu d_{x_0}\le(1-\mu)d_x$.

$$\begin{aligned}
\|D_{ij}u\|^{(2)}_{0,\alpha;B} &\le \sup_{x\in B} d^2|D_{ij}u| + \sup x,y\in B d_{x,y}^{2+\alpha}\frac{|D_{ij}u(x)-D_{ij}u(y)|}{|x-y|^\alpha} \\
&\le \frac{\mu^2}{(1-\mu)^{2+\alpha}}\left(\sup_{x\in\Omega} d_x^2|D_{ij}u| + \sup x,y\in\Omega d_{x,y}^{2+\alpha}\frac{|D_{ij}u(x)-D_{ij}u(y)|}{|x-y|^\alpha}\right) \\
&\le \frac{\mu^2}{(1-\mu)^{2+\alpha}}|D_{ij}u|^{(2)}_{0,\alpha;\Omega} \\
&\le \frac{\mu^2}{(1-\mu)^{2+\alpha}}\|u\|^{(*)}_{2,\alpha;\Omega} \le 8\mu^2\|u\|^{(*)}_{2,\alpha;\Omega}
\end{aligned}$$

$$
\begin{aligned}
\|D_i u\|^{(1)}_{0,\alpha;B} &\leq \sup_{x\in B} d|D_i u| + \sup_{x,y\in B} d^{1+\alpha}_{x,y} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha} \\
&\leq \frac{\mu}{(1-\mu)^{1+\alpha}} \left(\sup_{x\in\Omega} d_x |D_i u| + \sup_{x,y\in\Omega} d^{1+\alpha}_{x,y} \frac{|D_i u(x) - D_i u(y)|}{|x-y|^\alpha} \right) \\
&\leq \frac{\mu}{(1-\mu)^{1+\alpha}} \|D_i u\|^{(1)}_{0,\alpha;\Omega} \\
&\leq 4\mu \|D_i u\|^{(1)}_{0,\alpha;\Omega} \leq 4\mu \|u\|^{(*)}_{1,\alpha;\Omega}
\end{aligned}
$$

$$
\begin{aligned}
\|u\|^{(0)}_{0,\alpha;B} &\leq \frac{1}{(1-\mu)^\alpha} \|u\|^{(0)}_{0,\alpha;\Omega} \\
&\leq 2\|u\|^{(*)}_{0,\alpha;\Omega}
\end{aligned}
$$

The coefficients:

$$
\begin{aligned}
\|a^{ij}(x_0) - a^{ij}(x)\|^{(0)}_{0,\alpha;B} &\leq \sup_{x\in B} |a^{ij}(x_0) - a^{ij}(x)| + \sup_{x,y\in B} d^\alpha \frac{|a^{ij}(x) - a^{ij}(y)|}{|x-y|^\alpha} \\
&\leq \sup_{x\in B} |x_0 - x|^\alpha \frac{|a^{ij}(x_0) - a^{ij}(x)|}{|x-x_0|^\alpha} \\
&\qquad + \frac{\mu^\alpha}{(1-\mu)^\alpha} \sup_{x,y\in\Omega} d^\alpha_{x,y} \frac{|a^{ij}(x) - a^{ij}(y)|}{|x-y|^\alpha} \\
&\leq \sup_{x\in B} d^\alpha \frac{|a^{ij}(x_0) - a^{ij}(x)|}{|x-x_0|^\alpha} + [a^{ij}]^{(*)}_{0,\alpha;\Omega} \\
&\leq 2\, \frac{\mu^\alpha}{(1-\mu)^\alpha} \left([a^{ij}]_{0,\alpha;\Omega}\right) \\
&\leq 2^{1+\alpha} \mu^\alpha [a^{ij}]^{(*)}_{0,\alpha;\Omega} \\
&\leq 4\Lambda\mu^\alpha
\end{aligned}
$$

$$
\begin{aligned}
\|b^i\|^{(1)}_{0,\alpha;B} &\leq \sup_{x\in B} d|b^i| + \sup_{x,y\in B} d^{1+\alpha} \frac{|b^i(x) - b^i(y)|}{|x-y|^\alpha} \\
&\leq \frac{\mu}{(1-\mu)^{1+\alpha}} \left(\sup_{x\in B} d_x |b^i| + \sup_{x,y\in B} d^{1+\alpha}_{x,y} \frac{|b^i(x) - b^i(y)|}{|x-y|^\alpha} \right) \\
&\leq 4\mu \|b^i\|^{(1)}_{0,\alpha;\Omega} \leq 2\mu^\alpha \Lambda
\end{aligned}
$$

$$\|c\|^{(2)}_{0,\alpha;B} \leq \frac{\mu^2}{(1-\mu)^{2+\alpha}} \|c\|^{(2)}_{0,\alpha;\Omega}$$
$$\leq 8\mu^2 \|c\|^{(2)}_{0,\alpha;\Omega}$$

Putting all this together, we have

$$\|F\|^{(2)}_{0,\alpha;B} \leq$$

□

4. The estimates we took for granted

Here we prove the estimates that we took for granted.

4.1. We start with (1.1), which says

$$D_{ij}w(x) = \int_{B_2} D_{ij}\Gamma(x-y)(f(y)-f(x))dy - f(x)\int_{\partial B_2} D_i\Gamma(x-y)\nu_j(y)ds_y$$

For (1.1) to be true, we actually need some additional hypotheses, which are satisfied in the statement of Lemma2.1. We need f to be a bounded, locally C^α function. If f is a bounded, locally C^α on some arbitrary domain, Ω, then the identity still holds for Ω_0 instead of B_2, where $\Omega \subset \Omega_0$ where the divergence theorem holds, and f vanishes outside of Ω. We prove this for the arbitrary domain, Ω.

Proof. First we need to prove the following identity.

Claim 1.

$$D_i w(x) = \int_\Omega D_i\Gamma(x-y)f(y)dy$$

Proof. The function

$$v(x) = \int_\Omega D_i\Gamma(x-y)f(y)dy$$

is well-defined by (1.2).
Let $\eta_\epsilon(x) \in C^\infty(\mathbb{R})$ be such that
(1) $0 \leq \eta_\epsilon(x) \leq 1$ (2) $\eta_\epsilon(x) = 0$ for $x \leq \epsilon$ (3) $\eta_\epsilon(x) = 1$ for $x \geq 2\epsilon$ (4) $|\nabla\eta_\epsilon| \leq \frac{2}{\epsilon}$
Define $w_\epsilon(x)$ to be

$$w_\epsilon(x) = \int_\Omega \Gamma(x-y)f(y)dy$$

Then $w_\epsilon(x) \in C^1$ and $w_\epsilon(x) \to w$ uniformly.
So we have

$$
\begin{aligned}
v(x) - D_i w_\epsilon(x) &= \int \Omega(D_i\Gamma(x-y) - D_i(\Gamma(x-y)\eta_\epsilon(|x-y|)))f(y)dy \\
&= \int_\Omega D_i((1-\eta_\epsilon(|x-y|))\Gamma(x-y))f(y)dy \\
&= \int_{|x-y|\leq 2\epsilon} D_i((1-\eta_\epsilon(|x-y|))\Gamma(x-y))f(y)dy
\end{aligned}
$$

We have

$$
\begin{aligned}
|v(x) - D_i w_\epsilon(x)| &\leq |f|_0 \int_{|x-y|\leq 2\epsilon} (\frac{2}{\epsilon}|\Gamma(x-y)| + |D_i\Gamma(x-y)|)dy \\
&\leq |f|_0 \left(\frac{2}{\epsilon}\int_{|z|\leq 2\epsilon} |\Gamma(z)|dz + \int_{|z|\leq 2\epsilon} |D_i\Gamma(z)|dz\right) \\
&\leq C|f|_0 \left(\frac{2}{\epsilon}\int_{|z|\leq 2\epsilon} \frac{1}{|z|^{n-2}}dz + \int_{|z|\leq 2\epsilon} \frac{1}{|z|^{n-1}}dz\right) \\
&\leq C|f|_0 \left(\frac{2}{\epsilon}\int_{r\leq\epsilon} rdr + \int_{r\leq\epsilon} dr\right) \\
&\leq C|f|_0\epsilon
\end{aligned}
$$

Thus, $D_i w_\epsilon \to v$ uniformly on compact subsets as $\epsilon \to 0$. Thus, $w(x) \in C^1$ and $D_i w = v$. This gives us Claim 1. □

Now,

$$
u(x) = \int_{\Omega_0} D_{ij}\Gamma(x-y)(f(y) - f(x))dy - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j ds_y
$$

is well defined by virtue of (1.2).

Let

$$
v_\epsilon(x) = \int_\Omega D_i\Gamma(x-y)\eta_\epsilon(|x-y|)f(y)dy
$$

Now $v_\epsilon \in C^1(\Omega)$, and $v_\epsilon \to D_i w$ uniformly by the above claim.

Also, we have

$$
\begin{aligned}
D_j v_\epsilon(x) &= \int_\Omega D_j(D_i\Gamma(x-y)\eta_\epsilon(|x-y|))f(y)dy \\
&= \int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\epsilon(|x-y|))(f(y)-f(x))dy \\
&\qquad + f(x)\int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\epsilon(|x-y|))dy \\
&= \int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\epsilon(|x-y|))(f(y)-f(x))dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\eta_\epsilon(|x-y|)\nu_j(y)ds_y \\
&= \int_{\Omega_0} D_j(D_i\Gamma(x-y)\eta_\epsilon(|x-y|))(f(y)-f(x))dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)ds_y
\end{aligned}
$$

We have

$$
\begin{aligned}
|u(x) - D_j v_\epsilon(x)| &\leq \int_{|x-y|\leq 2\epsilon} D_j(1-\eta_\epsilon)D_i\Gamma(f(y)-f(x))dy \\
&\leq [f]_\alpha \int_{|x-y|\leq 2\epsilon} (|D_{ij}\Gamma| + \frac{2}{\epsilon}|D_i\Gamma|)|x-y|^\alpha dy \\
&\leq C[f]_\alpha \left(\int_{|x-y|\leq 2\epsilon} \frac{1}{|x-y|^{n-\alpha}}dy + \frac{2}{\epsilon}\int_{|x-y|\leq 2\epsilon} \frac{1}{|x-y|^{n-1-\alpha}}dy \right) \\
&\leq C[f]_\alpha((2\epsilon)^\alpha + \frac{2}{\epsilon}(2\epsilon)^{1-\alpha}) \\
&\leq C[f]_\alpha \epsilon^\alpha
\end{aligned}
$$

Thus $D_j v_\epsilon(x)$ converges to u uniformly on compact subsets as $\epsilon \to 0$. But $v_\epsilon \to D_i w$, so $w \in C^2(\Omega)$ and $u = D_{ij}w$.

□

4.2. Next, we estimate the first and second derivatives of fundamental solution of Laplace's equation (1.2), and also esimates for the third derivative used in bounding the integral I_6.

The fundamental solution of Laplace's equation for $n > 2$ is

$$
\Gamma(x-y) = \frac{1}{n(2-n)\omega_n}|x-y|^{2-n}
$$

Differentiating once and then again, we have

$$
\begin{aligned}
D_i\Gamma(x-y) &= \frac{1}{n\omega_n}(x_i - y_i)|x-y|^{-n} \\
D_{ij}\Gamma(x-y) &= \frac{1}{n\omega_n}\{|x-y|^2\delta_{ij} - n(x_i-y_i)\}|x-y|^{-n-2}
\end{aligned}
$$

Differentiating a third time, we have

$$D_{ijk}\Gamma(x-y) = \frac{1}{n\omega_n}\{-n|x-y|^{-n-2}(x_k-y_k)\delta_{ij} - n(x_j-y_j)|x-y|^{-n-2}\delta_{ik}$$
$$- n(x_i-y_i)|x-y|^{-n-2}\delta_{jk}$$
$$- n(x_i-y_i)(x_j-y_j)(x_k-y_k)|x-y|^{-n-4}\}.$$

Thus, we have the following estimates:

$$|D_i\Gamma(x-y)| \leq \frac{1}{n\omega_n}|x-y|^{1-n}; \tag{4.1}$$

$$|D_{ij}\Gamma(x-y)| \leq \frac{1}{\omega_n}|x-y|^{-n}; \tag{4.2}$$

$$|D_{ijk}\Gamma(x-y) \leq \frac{n+5}{\omega_n}|x-y|^{-n-1}. \tag{4.3}$$

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