An Investigation of an Irrationality Proof

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1 Introduction

The discovery of the first and perhaps most popular irrational number is credited to the Pythagorean scholar, Hippasus. The story goes that using geometric methods, Hippasus first proved the irrationality of $\sqrt{2}$ upon a boat with Pythagoras. The irrationality of numbers was so intolerable to Pythagoras that he threw the poor Hippasus overboard. Pythagoras believed "numbers rule the world," yet could not accept the existence of irrational numbers nor could he disprove their existence. Irrational numbers were finally accepted by the sixteenth century and in 1737, Euler proved the irrationality of e.

G.H. Hardy wrote in A Mathematician's Apology [3], that "The seriousness of a mathematical theorem lies, not in its practical consequences, which are negligible, but in the significance of the mathematical ideas which it connects." The fact that e is irrational is not that significant. The true value lies instead in its proof.

Euler used the series expansion of e to prove its irrationality. More pertinent, perhaps, is that every real number has a series representation. Consider the usual base 10 decimal expansion of a real number. The number could be represented as the sum of its integral part and its decimal part in base 10.

For example,

$$\pi = \sum_{n=0}^{\infty} \frac{a_n}{10^n},$$

where $a_0 = 3, a_1 = 1, a_2 = 4$, and a_i is equal to the integer in the 10^i decimal place of π. Moreover, the choice of base 10 is arbitrary. Any real number could be expressed as a series in any integral base.

This note investigates Euler's proof that e is irrational and tries to understand the full power of the proof technique, hoping to extend the ideas of the proof to general theorems about the irrationality of numbers. Along the way we describe other numbers where the same proof can be used to establish irrationality. Our goal is to try to describe all numbers that can be proved to be irrational using this proof technique.

2 Euler's proof of irrationality of e

We begin by giving the standard and most elementary proof that e is irrational. The proof of this section is the focus of the remainder of the note so we lay out the proof with great detail.

Theorem 2.1. *e is irrational.*

Proof. To prove that e is irrational we make use of the fact that the Taylor expansion for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1}$$

Thus letting $x = 1$ we have

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots .$$

Now assume, for a contradiction, that $e = \frac{p}{q}$ for some relatively prime integers p and q. Thus we have

$$\frac{p}{q} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Multiplying the series representation for e by $q!$ we have

$$(q-1)!p = \sum_{n=0}^{q} \frac{q!}{n!} + \sum_{n=q+1}^{\infty} \frac{q!}{n!}.$$

The first sum on the right hand side is a sum of integers. We call the second sum on the right hand side the "remainder sum". Since the left hand side and the first sum on the right are both integers we know that the remainder sum must be an integer. We will obtain a contradiction by showing that the remainder sum is bigger than 0, but less than 1. Therefore it cannot be an integer.

Our remainder sum is

$$\begin{aligned}
\sum_{n=q+1}^{\infty} \frac{q!}{n!} =& \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \\
<& \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots \\
=& \frac{1/(q+1)}{1-1/(q+1)} \\
=& \frac{1}{q}.
\end{aligned}$$

Using $q \geq 1$, we obtain a contradiction because our remainder sum must be an integer between 0 and 1.

□

3 Steps in the Proof of Irrationality of e

To understand this proof we will begin by trying to understand what we needed in order to prove the irrationality of e. We begin in this section by understanding what properties of e that allowed the proof technique to work. We outline the "steps" in the proof and then in the following sections we try to construct numbers that this proof technique can be applied to in order to prove irrationality.

1. Begin with a number α which has a corresponding infinite sum of the form
$$\alpha = \sum_{n\geq 0} \frac{1}{a_n},$$
where the a_n are integers.

2. Assume for a contradiction your numbers is of the form $\frac{p}{q}$ where p and q are relatively prime.

3. Multiply $\frac{p}{q} = \sum_{n\geq 0} \frac{1}{a_n}$ by an integer that has q as a factor and leaves us with two sums on the right, one which is an integer and a second, which is small in size. We call the small sum the "remainder sum". The remainder sum should be small so that we can obtain a contradiction, by showing that it is supposed to be an integer, however it is between 0 and 1.

4. Show that the remainder sum must be between 0 and 1.

5. Conclude that the number is irrational by contradiction.

The substance of this proof technique lies in steps 3 and 4. As it is not obvious what integer we should multiply by in the third step, we explore this integer in more detail in the next section.

4 A First attempt at a general theorem

As notation we let $[c_1, c_2, \ldots, c_n]$ be the least common multiple of $c_1, c_2, \ldots, c_n$.

We try to use the steps outlined above to describe the general setting for the application of this proof technique. The first item that is up for interpretation in our steps is what "integer that has q has a factor" to multiply our sum by. We may write this integer as $I_q = qI$ for $I \in \mathbb{Z}$ since we know that we want q to be a factor of I_q. The hope is that this integer, when multiplied by the sum will break the sum into two pieces: one which is small and one which is an integer. More precisely we want

$$I_q \sum_{n\geq 1} \frac{1}{a_n} = \sum_{n=1}^{N} \frac{I_q}{a_N} + \sum_{n=N+1}^{\infty} \frac{I_q}{a_n},$$

where the first sum is an integer and the second sum is small. (Strictly speaking we do not need to split the sum in this way, but for simplicity we do this to attempt a general theorem.) Thus we have two conditions to try to impose. We begin by imposing the first, requiring that the first sum be an integer, and then we see whether or not the second sum is small.

One way to guarantee that the first sum is an integer is by requiring that $\frac{I_q}{a_n}$ for $n = 1, 2, \ldots, N$ is an integer. The easiest way to assure that we have this condition is by taking $I_q = qI = q[a_1, \ldots, a_N]$. We point out that this is not the most "optimal" way to choose I_q. By taking $I_q = [q, a_1, \ldots, a_N]$ we would be choosing I_q as small as possible while still guaranteeing that I_q/a_j will be an integer for $j = 1, \ldots, N$. However, for simplicity, we will assume $I_q = q[a_1, \ldots, a_N]$. Because $q[a_1, \ldots, a_N]$ and $[q, a_1, \ldots, a_N]$ differ by at most a factor of q we will see that we will not loose that much by taking I_q as we do. Returning to our problem, we see that with this choice of I_q we wish to have

$$q[a_1, \ldots, a_N] \sum_{n=N+1}^{\infty} \frac{1}{a_n} < 1.$$

Equivalently, in order to use this proof technique we must have

$$[a_1, \ldots, a_N] \sum_{n=N+1}^{\infty} \frac{1}{a_n} < \frac{1}{q}.$$

The problem now is that we must be able to obtain this inequality for all q, since we must obtain a contradiction for an arbitrary $\frac{p}{q}$.

If we knew that for all ε we could find an N such that

$$[a_1, \ldots, a_N] \sum_{n=N+1}^{\infty} \frac{1}{a_n} < \varepsilon,$$

then we would be able to obtain the desired contradiction.

With this argument in mind, we give the following theorem.

Theorem 4.1. *Let* $\alpha = \sum_{n\geq 0} \frac{1}{a_n} < \infty$ *with* $a_j \in \mathbb{N}$ *for all* j*. Furthermore, suppose that for all* ϵ *there exists* N *such that*

$$[a_1, a_2, \ldots, a_N] \sum_{n\geq N+1} \frac{1}{a_n} < \epsilon.$$

Then α *is irrational.*

Proof. The proof uses the steps we outlined in section 3. Since the technique is the focus of this paper we rewrite this argument in order to illustrate precisely how we are using the technique.

The number α has a representation as an infinite sum. Assume for a contradiction that $\alpha = \frac{p}{q}$. Now by assumption we know there is an N such that

$$[a_1, a_2, \ldots, a_N] \sum_{n\geq N+1} \frac{1}{a_n} < \frac{1}{q}.$$

Then we multiply α by $q[a_1, \ldots, a_N]$ so that

$$q[a_1, \ldots, a_N]\alpha = p[a_1, \ldots, a_N] = q \sum_{n=0}^{N} \frac{[a_1, \ldots, a_N]}{a_n} + q[a_1, \ldots, a_N] \sum_{n=N+1}^{\infty} \frac{1}{a_n}.$$

We know that $p[a_1, \ldots, a_N]$ and $q \sum_{n=0}^{N} \frac{[a_1,\ldots,a_N]}{a_n}$ are both integers so that $q[a_1, \ldots, a_N] \sum_{n=N+1}^{\infty} \frac{1}{a_n}$ must also be an integer, but it cannot be since by assumption $0 < q \sum_{n=N+1}^{\infty} \frac{a_N}{a_n} < 1$. □

The irrationality of e is a consequence of this theorem, but we can also prove the irrationality of many other numbers using this result, including $\sum_{n\geq 1} \frac{1}{7^{n^2}}$. More generally it allows us to prove the irrationality of $\sum_{n\geq 1} \frac{1}{c^n b^{n^2}}$, where $c \geq 1$ and $b \geq 2$. We give this particular example as a corollary.

Corollary 4.2. *Let* $c, b, s \in \mathbb{N}$ *with* $|b| > 1$ *and* $s > 1$ *an integer. Let*

$$\alpha = \sum_{n\geq 1} \frac{1}{c^n b^{n^s}}.$$

Then α *is irrational.*

Proof. We wish to apply Theorem 4.1 with $a_n = c^n b^{n^s}$. We see that $[a_1, \ldots, a_n] = c^n b^{n^s}$. The result will follow we can prove that for all ϵ there exists N such that

$$\sum_{n \geq N+1} \frac{c^N b^{N^s}}{c^n b^{n^s}} = \sum_{n \geq N+1} \frac{1}{c^{n-N} b^{(n^s - N^s)}} < \epsilon.$$

To establish this claim notice that for N sufficiently large, since $s \geq 2$, $\frac{b^{N^s}}{b^{n^s}} \leq \frac{b^{N^2}}{b^{n^2}}$ we have

$$\sum_{n \geq N+1} \frac{1}{c^{n-N} b^{(n^s - N^s)}} \leq \sum_{n \geq N+1} \frac{1}{c^{n-N} b^{(n^2 - N^2)}}$$

Using this we have

$$\begin{aligned}
\sum_{n \geq N+1} \frac{1}{c^{n-N} b^{(n^s - N^s)}} &\leq \sum_{n \geq N+1} \frac{1}{c^{n-N} b^{(n-N)(n+N)}} \\
&= \sum_{k \geq 1} \frac{1}{c^k b^{k(k+2N)}} \\
&< \frac{1}{b^{2N}} \sum_{k \geq 1} \frac{1}{c^k b^{k^2}} \\
&< \frac{1}{b^{2N}} \sum_{k \geq 1} \frac{1}{c^k b^k} \\
&= \frac{1}{b^{2N}} \frac{cb}{cb - 1}.
\end{aligned}$$

Now we may choose N so large that for any $\epsilon > 0$, $b^{2N} > \frac{ab}{\epsilon(ab-1)}$. Therefore for any ϵ we can obtain the desired inequality. Then applying Theorem 4.1 we see that α is irrational. $\square$

While our theorem allows us to prove the irrationality of many numbers, one may wonder whether it allows us to prove the irrationality of any irrational number. An affirmative answer to this would be too much to hope for. On the other hand, our theorem may appear a bit technical since it involves some careful estimates.

There are other methods of noticing the irrationality of certain sums of this form. Consider the number $\sum_{n \geq 1} \frac{1}{10^{n^2}}$, which is of this form with $a = 1$,

$b = 10$, and $s = 2$. Our Corollary applies here, but we may use more elementary methods to observe that this number is irrational. A number is rational if and only if its decimal expansion has a finite number of non-zero terms or if it eventually repeats. Using this fact, it is easy to see that

$$\sum_{n\geq 1} \frac{1}{10^{n^2}} = \frac{1}{10} + \frac{1}{10000} + \cdots$$

cannot have a decimal expansion which repeats, nor does it have only finitely many non-zero digits. Thus it is irrational.

For this particular example, it would be troublesome to use our theorem. However, no such simple argument can be used to prove the irrationality of e.

The main condition of this theorem

$$[a_1, a_2, \ldots, a_N] \sum_{n\geq N+1} \frac{1}{a_n} < \epsilon$$

is a bit mysterious. Furthermore, this condition still involves checking something about an infinite sum and as a result seems a bit unwieldy. Equivalently, we set

$$\varepsilon_N := \frac{\varepsilon}{[a_1, \ldots, a_N]},$$

then we require

$$\sum_{n=N+1}^{\infty} \frac{1}{a_n} < \varepsilon_N. \tag{2}$$

We were able to establish the irrationality of these sums due to the fact that the "remainder sums", $\sum_{n\geq N+1} \frac{1}{a_n}$, could be made really small without needing to pick N too large. This is necessary, since if we pick N larger and larger then the ε_N became smaller and smaller. This observation leads us to see that what we really needed was for these infinite sums to be rapidly convergent in some sense, where the the speed at which it converges is dictated by how quickly the least common multiple of $a_1, \ldots, a_N$ grows. We will explore this phenomenon in the next section.

5 Rapidly Convergent Sums

We return to the discussion of the proof technique in the previous section to better understand the necessary behavior of infinite sums in order to employ our proof technique.

Notice there is a balance taking place. We must choose N large enough so that $\sum_{n\geq N+1}\frac{1}{a_n}$ is small, since however larger N is, the smaller ε_N gets. The rate at which ε_N decreases is controlled by the rate at which $[a_1,\ldots,a_N]$ increases, and the rate at which $\sum_{n\geq N+1}\frac{1}{a_n}$ decreases is determined by how quickly the a_n's grow. So we want the a_n's to grow quickly and the $[a_1,\ldots,a_N]$ to grow at a slow rate. In other words, we need the a_N to have many divisors in common with the a_j's for $j < N$, in order to insure that $[a_1,\ldots,a_N]$ does not grow more quickly than it needs to. Since we want $[a_1,\ldots,a_N]$ to grow slowly and knowing that $[a_1,\ldots,a_N]\geq a_N$, the best we can hope for is that

$$[a_1,\ldots,a_N]=a_N. \tag{3}$$

The most obvious example of a sequence $(a_n)_{n=1}^{\infty}$ where a_n has many divisors in common with the preceding a_j for $j < n$ and for which the a_n increases is the sequence $a_n = b^n$ for some $b\in\{2,3,4,\cdots\}$. In fact, this example is perhaps the best behaved sequence we can ask for, since this sequence satisfies the condition given in equation (3).

But in this case we see that we have a geometric series and

$$\sum_{n\geq 1}^{\infty}\frac{1}{b^n}=\frac{1}{b-1}\in\mathbb{Q}.$$

The problem with this sequence is that it did not grow quickly enough. If we make this sequence grow more quickly by considering $a_n = b^{n^2}$, we maintain the condition given by equation (3), and we know by Corollary 4.2 that the sum $\sum_{n\geq 1}\frac{1}{a_n}$ is an irrational number.

Moreover, returning to the original example of e, we have $a_n = n!$ and so $[a_1,\ldots,a_N] = a_N = N!$. We satisfy condition (3), and here we are able to exhibit the irrationality of the sum!

We have seen a couple examples of sequences $(a_n)_{n\geq 1}$ that satisfy condition (3) and result in irrational numbers $\sum_{n\geq 1}\frac{1}{a_n}$ and one example of such a sequence $(a_n)_{n\geq 1}$ that resulted in a rational number. What then are sufficient conditions on the growth of the a_n so that we can prove the irrationality of α? Next we will consider the rate of growth of the a_n when they satisfy condition (3).

6 Rate of Growth of a_n

In this section we suppose that the sequence of integers satisfies $[a_1,\ldots,a_N] = a_N$ for all $N \geq 1$. Since we assume that the a_n grow with n and that they are integers we know that $a_n = [a_1,\ldots,a_{n-1},a_n] = k[a_1,\ldots,a_{n-1}] = ka_{n-1}$. Since the a_n must grow and are integers we see that $k \geq 2$. Therefore, we establish by induction that $a_n \geq 2^n$. So we see that the a_n must grow at least exponentially when equation (3) is satisfied.

In the example above, we saw that simple exponential growth with $a_n = b^n$ was not enough to establish irrationality. So it is natural to ask: How much faster than exponential growth is necessary so that for all $\varepsilon > 0$ we can find a N so that

$$\sum_{n\geq N+1}\frac{1}{a_n} < \frac{\varepsilon}{a_N}?$$

Since we have exponential growth already, we may ask about a sequence a_n which grows like $b^{nL(n)}$ for some strictly increasing function $L:\mathbb{N}\to\mathbb{R}$ and some number $b > 1$. For example, we might consider $L(n) = n^\delta$ with $\delta > 0$ or $L(n) = \log(n)$.

With this assumption on the growth of the a_n we ask whether for all $\varepsilon > 0$ does there exist an N such that

$$\sum_{n\geq N+1}\frac{1}{b^{nL(n)}} < \frac{\varepsilon}{b^{NL(N)}}.$$

To answer this question consider the following calculation:

$$\sum_{n\geq N+1} \frac{1}{b^{nL(n)}} = \sum_{k=1}^{\infty} \frac{1}{b^{(k+N)L(k+N)}} < \frac{1}{b^{NL(N)}} \sum_{k=1}^{\infty} \frac{1}{b^{kL(N)}} = \frac{1}{b^{NL(N)}(b^{L(N)}-1)},$$

where the inequality follows because $L(k+N) > L(k)$ and $L(k+N) > L(N)$ for all N. Therefore, to have the desired estimate for the "remainder sum" it is enough to have

$$\frac{1}{b^{NL(N)}(b^{L(N)}-1)} < \frac{\varepsilon}{b^{NL(N)}}.$$

Because $L(N)$ is strictly increasing and $b > 1$, we can always choose a large enough N to do so.

This calculation shows that if a_n is growing at a rate similar to $b^{nL(n)}$ with L some strictly increasing function and $[a_1, \ldots, a_n] = a_n$ for all n, then $\alpha = \sum_{n\geq 1} 1/a_n$ will be an irrational number. This leads us to a second general theorem.

Before we can precisely state the theorem, we will have to define what we mean by a_n is similar to $b^{nL(n)}$. We define the notion of asymptotic.

Definition 6.1. For two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ we write $a_n \sim b_n$ if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

As a result we know that for all $\epsilon > 0$ there exists an N such that for all $n \geq N$, we have $|\frac{a_n}{b_n} - 1| < \epsilon$. Thus $(1-\epsilon)b_n < a_n < (1+\epsilon)b_n$.

We can now give the precise theorem.

Theorem 6.2. *Suppose $(a_n)_{n=1}^{\infty}$ is an increasing sequence of positive integers such that*

1. $[a_1, \ldots, a_n] = a_n$ *for all* n
2. $a_n \sim Cb^{nL(n)}$ *for some* $b > 1$, $C > 0$, *and* $L : \mathbb{N} \to \mathbb{R}$ *is a positive strictly increasing function for all* n,

then $\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n}$ *is an irrational number.*

Proof. For a contradiction assume that

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{p}{q}$$

for integers p and q. Then we see that for all $M \geq 1$,

$$qa_M \sum_{n=M+1}^{\infty} \frac{1}{a_n} = a_M p - qa_M \sum_{n=1}^{M} \frac{1}{a_n} \in \mathbb{Z}.$$

We will obtain a contradiction by finding an M such that

$$qa_M \sum_{n=M+1}^{\infty} \frac{1}{a_n} < 1.$$

By assumption we may choose N_1 large enough so that with $\epsilon = 1/2$, $C(1-\epsilon)b^{nL(n)} < a_n < C(1-\epsilon)b^{nL(n)}$ for all $n \geq N_1$. Additionally, because $b > 1$ and $L(n)$ is an eventually strictly increasing function in n, we may choose N_2 large enough so that $\frac{1}{b^{L(N_2)-1}} < \frac{1}{3q}$. Set $M = \max(N_1, N_2)$.

Now we have

$$\begin{aligned}
\sum_{n=M+1}^{\infty} \frac{a_M}{a_n} < \sum_{n=M+1}^{\infty} \frac{C(1+\epsilon)b^{ML(M)}}{C(1-\epsilon)b^{nL(n)}} \\
=& \frac{(1+\epsilon)}{(1-\epsilon)} \sum_{k=1}^{\infty} \frac{b^{ML(M)}}{b^{(M+k)L(M+k)}} \\
<& \frac{(1+\epsilon)}{(1-\epsilon)} \sum_{k=1}^{\infty} \frac{1}{b^{kL(M+k)}} \\
<& \frac{(1+\epsilon)}{(1-\epsilon)} \sum_{k=1}^{\infty} \frac{1}{b^{kL(M)}} \\
=& \frac{(1+\epsilon)}{(1-\epsilon)} \frac{1}{b^{L(M)}-1}.
\end{aligned}$$

To finish the proof we notice that for $\epsilon = 1/2$ we have $\frac{(1+\epsilon)}{(1-\epsilon)} = 3$, and that since $M \geq N_2$, we have $\frac{(1+\epsilon)}{(1-\epsilon)} \frac{1}{b^{L(M)-1}} < \frac{1}{q}$. $\square$

We note that we really only need the function L to be eventually strictly increasing, since we can deal with a finite number of terms separately. More precisely if L is increasing for all $n \geq N_0$ then we consider the two sums $\sum_{n=1}^{N_0} \frac{1}{a_n}$ and $\sum_{n>N_0} \frac{1}{a_n}$. The first sum is rational since it is the sum of rational numbers. While the second sum can be treated just as the sum considered in Theorem 6.2.

As an example of this theorem, let us return to $e = \sum_{n\geq 1} \frac{1}{n!}$. Then, using the famous Stirling's Approximation ([5] page 253) to the factorial we know that

$$n! \sim n^n e^{-n} \sqrt{2\pi n} = e^{n\log(n)} e^{-n} e^{\log(2\pi n)} = e^{n(\log(n)-1+\frac{\log(2\pi n)}{n})}.$$

To apply the theorem it remains to check that $\log(n) - 1 + \frac{\log(2\pi n)}{n}$ is an eventually strictly increasing function. To check this set $f(x) := \log(x) - 1 + \frac{\log(2\pi x)}{x}$. Then $f'(x) = \frac{1}{x^2}(x + 1 - \log(2\pi x))$. Now it is easy to verify that $x + 1 - \log(2\pi x) > 0$ for all $x \geq 1$. So in this case $L(n) = \log(n) - 1 + \log(2\pi n)/n$ satisfies the needed condition to apply Theorem 6.2.

It is natural to wonder if the condition $[a_1, \ldots, a_n] = a_n$ necessary? In other words, could we relax this condition and still use the same ideas that we discussed above?

The general principle here is that we have to maintain the delicate balance we discussed in Section 5. If we relax this condition a little bit, requiring a bit less, then we will need the a_n to grow a bit more quickly.

7 Summary and Exercises for Further Exploration

We have presented two "general" theorems which can be used to prove the irrationality of a number, represented by an infinite sum of the form $\sum_{n\geq 1} \frac{1}{a_n}$ with the $a_n \in \mathbb{N}$. The first theorem has a condition that seemed somewhat awkward because it involved checking a condition on an infinite sum. The second theorem gives us a more intuitive feel for the necessary behavior of the infinite sum in the proof of the irrationality of e. We compare the usefulness of the two theorems with the following examples.

7.1 Theorem 4.1 versus Theorem 6.2

Here we outline how to use each of our theorems to establish the irrationality of

$$\alpha_1 := \sum_{n\geq 1} \frac{1}{(2n+1)!!},$$

where

$$(2n+1)!! = (2n+1)(2n-1)\cdots 5\cdot 3\cdot 1$$

is the double factorial.

This exercise shows how we can use Theorem 4.1 to prove that α_1 is irrational.

Exercise 7.1. Use $[1!!,\ldots,(2N+1)!!] = (2N+1)!!$ and

$$\begin{aligned}\sum_{n\geq N+1} \frac{(2N+1)!}{(2n+1)!} =& \frac{1}{(2N+3)} + \frac{1}{(2N+3)(2N+5)} + \cdots \\ <& \frac{1}{(2N+3)} + \frac{1}{(2N+3)^2} + \frac{1}{(2N+3)^3} + \cdots \\ =& \frac{1}{2N+2}.\end{aligned}$$

with Theorem 4.1 to prove that α_1 is irrational.

The next four exercises show how Theorem 6.2 can be applied to prove the irrationality of α_1.

Exercise 7.2. Stirling's Formula gives the following

$$n! \sim n^n e^{-n}\sqrt{2\pi n}.$$

Use this to show that

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!} \sim \left(1+\frac{1}{2n}\right)^n (2n+1)^{(n+1)} \sqrt{\frac{2n+1}{n}} e^{-n-1}.$$

Exercise 7.3. We write the previous result as

$$(2n+1)!! \sim e^{-1/2} e^{n\, L_1(n)},$$

where

$$L_1(n) := \log(2n+1) - 1 + \frac{1}{2n}\left(3\log(2n+1) - \log(n)\right).$$

Exercise 7.4. Show that for all $n \geq 2$, $L_1(n)$ is increasing with n. (Hint: You may want to consider $L_1(x)$ where x is a real variable and show that the derivative of L_1 is positive for all $x \geq 2$.)

Exercise 7.5. Use Exercise 7.4 and Theorem 6.2 to prove that α_1 is irrational.

Exercises 7.1 - 7.5 show how Theorem 4.1 gives us a nice clean proof of the irrationality of α_1, while using Theorem 6.2 to prove the irrationality of α_1 is much more cumbersome. The following exercises show how Theorem 6.2 can be more suited for proving the irrationality of a set of numbers.

Exercise 7.6. Bailey and Crandall consider the following [1] numbers

$$\alpha_{b,c} = \sum_{n\geq 1} \frac{1}{c^n\, b^{c^n}}.$$

They show that not only are these numbers irrational, but they are transcendental and normal! Their work has applications to random number generators.

Use Theorem 6.2 to show that $\alpha_{b,c}$ is irrational for $b, c \geq 2$ by noticing that

$$c^n b^{c^n} = b^{n\log_b(c) + c^n} = b^{n(\log_b(c) + \frac{c^n}{n})},$$

and showing that

$$L_{b,c}(n) := \log_b(c) + \frac{c^n}{n}$$

is eventually strictly increasing.

Here $\log_b(c)$ is the logarithm of c taken to the base b.

The Upshot

While the first of our general theorems is not very illuminating since it involves a condition about an infinite sum, it is still useful in application. On the other hand, our second general theorem gives us more of an intuition for the main idea behind the proof technique. As a general principal, in light of Theorem 6.2 it is worthwhile to consider the growth of a_n and check that it is faster than linear exponential before attempting to use this technique to prove that a number of the form $\alpha = \sum_{n\geq 1} \frac{1}{a_n}$ is irrational. However, in the actual proof of the irrationality of α, Theorem 4.1 may be much less cumbersome.

7.2 Non-examples

In the proof that e was irrational we began with the Taylor series for e^x. It is natural, then, to start with the Taylor series for a different function, f, and try to prove that $f(n)$ is irrational for some value of n. For example, we may begin with the Taylor series for

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

For $b \in \mathbb{N}$ we are led naturally to consider

$$\alpha_b := \sum_{n\geq 1} \frac{1}{b^n n} = -\log\left(1 - \frac{1}{b}\right).$$

It is known that such α_b are irrational. With this in mind, we may hope to apply our theorems for some b. Unfortunately, neither of our theorems applies. Our second theorem does not apply since we do not have the $[a_1, \ldots, a_n] = a_n$ property nor we do not have $a_n \sim c^{nL(n)}$, for any increasing L. Thus, a more sophisticated method of proof is needed to show that these numbers are irrational.

To see that the proof technique for e cannot prove the sum $\sum_{n\geq 1}\frac{1}{b^n n}$ irrational, begin by noticing that for a give ϵ we hope to be able to find an N such that

$$[b, 2b^2, 3b^3, \cdots, Nb^N]\sum_{n\geq N+1}\frac{1}{nb^n} < \epsilon.$$

We might expect that $[b, 2b^2, 3b^3, \cdots, Nb^N] = b^N[1, 2, 3, \ldots, N]$. This is indeed the case for many b and N. Additionally, it is a known fact that using the celebrated Prime Number Theorem on distribution of the primes,

$$[1, 2, 3, \cdots, N] \sim e^N.$$

So we would then need to be able to find an N with

$$(e\cdot b)^N\sum_{n\geq N+1}\frac{1}{nb^n} = e^N\sum_{k=1}^{\infty}\frac{1}{(N+k)b^k}$$

really small, but as N gets larger, e^N goes to infinity at a faster rate than the sum converges to 0. Thus we should not hope to use this proof technique to establish the irrationality of α_b.

Exercise 7.7. Find infinitely many values of N such that $[b, 2b^2, 3b^3, \cdots, Nb^N] = b^N[1, 2, 3, \ldots, N]$ when $b = 2$.

Exercise 7.8. Show that in the case $a_n = 2^n n$ there does not exist a constant c and a strictly increasing function $L : \mathbb{N} \to \mathbb{R}$ such that $a_n \sim c^{nL(n)}$.

As another non-example we give one more exercise.

Exercise 7.9. Peter Borwein [2] showed that for any integer $q > 1$ and rational number c such that $c \neq 0, -q^n$ for any $n \in \mathbb{N}$, the sum $\sum_{n\geq 1}\frac{1}{q^n+c}$ is irrational. Convince yourself that the methods discussed in this paper cannot be used to prove such numbers are irrational. Remember that when $c = 0$ we know that this sum represents a rational number.

7.3 Sums of the form $\sum_{n\geq 1} \frac{b_n}{a_n}$

We made a lot of choices while trying to derive our two theorems, so it is natural to ask whether these theorems tell the whole story of this proof technique. With a little bit of examination, we see that the answer is most certainly "no". For example, these theorems cannot be applied to the numbers

$$\sum_{n\geq 1} \frac{b_n}{a_n},$$

where the b_n are integers just as the a_n are. In these exercises we show that the techniques here can be used to prove the irrationality of various numbers of this form.

Before we give an example we might use the intuition that Theorem 6.2 that leads us to believe that we can use the proof technique so long as the b_n grow very slowly in comparison to the rate at which the a_n grow (which should exhibit faster than linear exponential growth). The easiest example to consider in this case is if the b_n don't grow at all. We treated the case where the b_n are constant with our main theorems. The next step is to consider b_n's that are bounded but non-constant.

Exercise 7.10. Modify the proof of Theorem 4.1 to prove the following proposition:

Proposition 7.11. *Let $\alpha = \sum_{n\geq 0} \frac{b_n}{a_n} < \infty$ with $a_j, b_j \in \mathbb{N}$ for all j. Furthermore, suppose that for all ϵ there exists N such that*

$$[a_1, a_2, \ldots, a_N] \sum_{n\geq N+1} \frac{1}{a_n} < \epsilon$$

and that $b_n < M$ for some $M > 0$ and all n. Then α is irrational.

Exercise 7.12. Use Proposition 7.11 to prove that $e^{-1} + 2e$ is an irrational number.

Exercise 7.13. Formulate and prove a version of Theorem 6.2 for numbers of the form $\alpha = \sum_{n\geq 0} \frac{b_n}{a_n}$ where $b_n < M$ for all n and some $M > 0$.

What if b_n grows as n grows? From the Taylor series for e^x given in equation (1), another example which we might discuss is proving e^2 or even e^r is irrational for any $r > 1$. In those cases, we would have $b_n = 2^n$ or r^n, which clearly grow. In 1761, Johann Heinrich Lambert actually proved a stronger statement than the irrationality of e; he proved that e^r is irrational for any rational number $r \neq 0$.

We also point out that if we could establish that e^r is irrational for any r, then we may deduce that $\log\left(\frac{b}{b-1}\right)$ is also irrational. Thus this would allow us to indirectly use our method to establish the irrationality of the α_b discussed in Section 7.2. Indeed, if $\log\left(\frac{b}{b-1}\right)$ were rational then we would have $\frac{p}{q} = \log\left(\frac{b}{b-1}\right)$ for some integers p and q, and thus

$$e^p = \left(e^{\frac{p}{q}}\right)^q = \left(e^{\log(b/(b-1))}\right)^q = \left(\frac{b}{b-1}\right)^q.$$

Thus contradicting the fact that e^p is irrational.

As one more exercise consider the following:

Exercise 7.14. Use the identity

$$\frac{\pi \log(2)}{2} = \sum_{n=0}^{\infty} \frac{(2n+1)!!}{n! 2^n (2n+1)^3}$$

to prove that $\pi \log(2)$ is irrational.

Acknowledgment We add our thanks here.

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