EXAMPLES OF KÄHLER MANIFOLDS

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Example 1 ($\mathbb{C}^n$)**.**

We begin with a somewhat silly example, but it's nonetheless important to exhibit that our most basic, nicest complex manifold is Kähler . Give $\mathbb{C}^n$ the standard hermitian metric, where $g_{i\bar{j}} = \delta_{i\bar{j}}$. The associated form,

$$\omega = \sum_i dz^i \wedge d\bar{z}^i$$

is closed since $\mathbb{C}^n$ is contractible and has no nontrivial higher cohomology. We call such a form Kähler .

Definition 1. Let g be a metric on a manifold M, and $\omega = \frac{i}{2}\sum_{i\bar{j}} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ its associated form. g is a **Kähler** metric if

$$d\omega = 0.$$

A Kähler manifold is one equipped with a Kähler metric.

Note. The constant of $\frac{i}{2}$ in the definition of ω is often left off in this and other notes.

Similarly, all complex manifolds X of dimension 1, that is Riemann surfaces, are Kähler . Any associated form ω is of degree 2 on X, which has no higher degree cohomology. Hence, on a Riemann surface, we must have $d\omega = 0$.

Example 2 ($\mathbb{C}P^n$)**.**

The complex projective space denoted by $\mathbb{C}P^n$ is $\mathbb{C}^n / \sim$, where $[z_0 : z_1 : \ldots : z_n] \sim \lambda[z_0 : z_1 : \ldots : z_n]$ for $\lambda \in \mathbb{C}$. Local charts are given by (U_i, ϕ_i) where $U_i = \{z \in \mathbb{C}P^n | z_i \neq 0\}$, and

$$\phi_i : [z_0, z_1, \ldots, z_n] \mapsto (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i})$$

These coordinates in $\mathbb{R}^n$ are denoted by $(w_1, w_2, \ldots, w_n)$. Transition maps ϕ_{ij} for $U_i \bigcap U_j$ is given by multiplication by $\dfrac{z_i}{z_j}$.

$\mathbb{C}P^n$ is a Kähler manifold with the Fubini-Study form given by

$$\omega_{FS} = \partial\bar{\partial}\log(\|z\|^2).$$

From this definition, it's not hard to see that $d\omega_{FS} = 0$. Locally, on a chart U_i, ω_{FS} is given by $\displaystyle\sum_{i,j} g_{i\bar{j}}\, dw^i \wedge d\bar{w}^j$, with

$$g_{i\bar{j}} = \frac{1}{(1+\|w\|^2)^2}\left\{(1+(\|w\|^2)\delta_{ij} - \bar{w}_i w_j\right\}.$$

In fact, $\mathbb{C}P^n$ is Kähler -Einstein.

Definition 2. A Kähler form ω is Einstein if for some constant c,

$$Ric(\omega) = c\omega.$$

A manifold is Kähler -Einstein if it admits a Kähler -Einstein form.

Locally, $Ric(\omega) = \sum_{i,j} R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ where $g^{i\bar{j}}$ is the $(i,j)^{th}$ entry of the matrix inverse to $[g_{i\bar{j}}]$ and $R_{i\bar{j}}$ is the trace on the first and third indices of the Riemannian curvature tensor. By a straightforward calculation from the properties and definition of the Riemannian curvature tensor (which we will not expand upon here), the Ricci curvature is also given by

$$Ric(\omega) = -\partial\bar{\partial}\log\det(g_{i\bar{j}}).$$

In the previous example, since $\mathbb{C}^n$ is flat, $Ric(\omega) = 0$, and so $\mathbb{C}^n$ is Kähler -Einstein.

We compute $\det(g_{i\bar{j}})$ for $\mathbb{C}P^n$. Note that the second part of the expression for $g_{i\bar{j}}$ are the entries of a projection matrix, that is projection onto the vector $\bar{\mathbf{w}}$. Thus, for the vector $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ (w_i the local coordinates at a point), choose orthonormal vectors $\{\mathbf{u_1}, \ldots \mathbf{u_{n-1}}\} \subset \mathbb{C}^n$ so that $\langle \bar{\mathbf{w}}, \mathbf{u_i}\rangle_{\mathbb{C}^n} = 0$. In particular, this implies that for such a $\mathbf{u} = [u_1, \ldots, u_n]$,

$$u_1 w_1 + \ldots u_n w_n = 0.$$

Normalize $\bar{\mathbf{w}}$ and let $\hat{w} = \dfrac{\bar{\mathbf{w}}}{\|\mathbf{w}\|}$ denote the normalized vector. With the basis $\{\hat{w}, u_1, \ldots, u_n\}$, our matrix $(g_{i\bar{j}})$ becomes

$$(g_{i\bar{j}}) = \begin{bmatrix} \frac{1}{(1+\|w\|^2)^2} & 0 & \cdot & \cdot & 0 \\ 0 & \frac{1}{(1+\|w\|^2)} & \cdot & \cdot & 0 \\ \vdots & 0 & & \ddots & \vdots \\ 0 & \cdot & & \cdot & \frac{1}{(1+\|w\|^2)} \end{bmatrix}.$$

All the entries should be clear based on our choice of vectors except for the diagonal entries which follows from a short straightforward computation. We have

$$\det(g_{i\bar{j}}) = \frac{1}{(1+\|w\|^2)^{n+1}}.$$

Hence,

$$\begin{aligned} Ric(\omega_{FS}) &= -\partial\bar{\partial}\log\det(g_{i\bar{j}}) \\ &= -\partial\bar{\partial}\log\frac{1}{(1+\|w\|^2)^{n+1}} \\ &= (n+1)\partial\bar{\partial}\log(\|z\|^2) \\ &= (n+1)\omega_{FS}. \end{aligned}$$

Example 3 (Smooth hypersurfaces of degree d in $\mathbb{C}P^n$)**.**

We start with some examples about line bundles on a manifold, M. Line bundles are determined by their transition maps. In particular, line bundles on $\mathbb{C}P^n$ are $\mathcal{O}(k)$, where the transition maps for $\mathcal{O}(k)$ is given by $g_{\alpha\beta} = \frac{z_\alpha^k}{z_\beta^k}$, where z^α and z^β are homogeneous coordinates on $\mathbb{C}P^n$ and the charts are given by $U_\alpha = \{z \in \mathbb{C}P^n | z_\alpha \neq 0\}$.

The **canonical bundle** K_M on a complex manifold of dimensional m is the determinant line bundle of holomorphic m-forms. That is, on a manifold M, let Ω_M denote the holomorphic tangent bundle. Then $\bigwedge^m \Omega_M$ is the canonical bundle. The canonical bundle on $\mathbb{C}P^n$ is $\mathcal{O}(-(n+1))$. Since $g_{i\bar{j}}$ is a metric on TM, we have that $\det(g_{i\bar{j}}$ is a metric on $-K_{\mathbb{C}P^n}$. Hence, the curvature form associated to $\det(g_{i\bar{j}})$ on $-K_M$ is the Ricci curvature of $g_{i\bar{j}}$. By a theorem of Chern, the Ricci curvature also represents the first Chern class $c_1(M)$ of the manifold M.

Let Z_F be a smooth hypersurface in $\mathbb{C}P^n$ of degree d, defined by the vanishing of the homogeneous polynomial $F(z)$ of degree d. Z_F is Kähler by the restriction of the Fubini-Study metric.

We are interested in whether Z_F has a positive, negative, or zero Chern class $c_1(M)$. Since Aubin and Yau had proven that for Kähler manifolds with negative and zero Chern class are in fact Kähler -Einstein, we are most interested in Kähler manifolds with positive first Chern class. Z_F sits as a smooth degree d hypersurface in $\mathbb{C}P^n$ and we would like to relate the canonical class of $\mathbb{C}P^n$ to that of the hypersurface. For that, we have the **Adjunction Formula**.

Theorem 3 (Adjunction Formula)**.** *Let V be a smooth hypersurface of degree d in a smooth variety M. Let K_M denote the canonical bundle on M, K_V denote the canonical bundle on the hypersurface and N_V the normal bundle in M on V. Then we have*

$$K_V = K_M\big|_V \otimes N_V.$$

For our example of Z_F, we have N_{Z_F} is given by $\mathcal{O}(d)|_{Z_F}$. Thus, by the Adjunction formula,

$$\begin{aligned} -K_{Z_F} &= (\mathcal{O}((n+1)) \otimes \mathcal{O}(-d))|_{Z_F} \\ &= \mathcal{O}(n+1-d)|_{Z_F} \end{aligned}$$

Since on $\mathbb{C}P^n$, sections of $\mathcal{O}(1)$ is generated by ω_{FS}, we have that the first Chern class of Z_F is represented by $(n+1-d)\omega_{FS}|_{Z_F}$. Thus, when d is sufficiently small, Z_F has a positive first Chern class.

In fact, we have an explicit formula for $Ric(\omega_{FS}|_{Z_F})$. The calculations aren't obvious, but they can be found in section 2.2 of Tian's *Canonical Metrics in Kähler Geometry.* As before, let $g_{i\bar{j}}$ denote the coefficients of the local expression for ω_{FS}, $[z_0 : \ldots : z_n]$ homogeneous coordinates on $\mathbb{C}P^n$. Over an open set where $z_0 \neq 0$, and $\frac{\partial F}{\partial z_n} \neq 0$,

$$\begin{aligned}
\det(g_{i\bar{j}}|_{Z_F}) &= \left\| \frac{\partial F}{\partial z_n} \right\|^{-2} \frac{\sum_{i=0}^{n} \|\frac{\partial F}{\partial z_i}\|^2}{(1+\|w\|^2)^n \|z_0\|^{2(d-1)}} \\
-\partial\bar{\partial} \log \det(g_{i\bar{j}}|_{Z_F}) &= -\partial\bar{\partial} \log \left(\frac{\sum_{i=0}^{n} \|\frac{\partial F}{\partial z_i}\|^2}{(1+\|w\|^2)^n \|z_0\|^{2(d-1)}} \right) \tag{0.1} \\
&\quad - \partial\bar{\partial} \log \left(\left\| \frac{\partial F}{\partial z_n} \right\|^{-2} \right) \\
&= -\partial\bar{\partial} \log \left(\frac{\sum_{i=0}^{n} \|\frac{\partial F}{\partial z_i}\|^2}{(1+\|w\|^2)^n \|z_0\|^{2(d-1)}} \right). \tag{0.2}
\end{aligned}$$

The last step follows from the fact that $\partial\bar{\partial} \log \|\frac{1}{z_0^{d-1}} \frac{\partial F}{\partial z_n}\|^2 = 0$, being the harmonic part of a holomorphic function.

Example 4 (Grassmannians)**.**

We consider a specific example for better understanding. Let $G(2,4)$ denote the Grassmannian of 2-planes in $\mathbb{C}^4$. Given such a plane, L, let $\{v_1, v_2\}$ be two independent vectors in $\mathbb{C}^4$ that generate L. Consider the map

$$\begin{aligned}
\Phi : &G(2,4) \to \mathbb{P}\left(\bigwedge^2 (\mathbb{C}^4) \right) \\
&L \longmapsto v_1 \wedge v_2
\end{aligned}$$

If we picked a different basis $\{w_1, w_2\}$ for L, then there exists an element A of GL_2 (the general linear group of rank 2) so that

$$A \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Thus, $w_1 \wedge w_2 = \det(A) v_1 \wedge v_2$, and they're the same element in $\mathbb{P}\left(\bigwedge^2(\mathbb{C}^4)\right)$. Hence, Φ is well defined as a map of sets. It's easy to see that from any totally decomposable vector in $\bigwedge^2(\mathbb{C}^4)$, we can recover the plane in $\mathbb{C}^4$. This is the **Plücker embedding**. To check that it's actually a subvariety, consider the map $\phi_\omega : \mathbb{C}^4 \to \bigwedge^3(\mathbb{C}^4)$ where $\omega \in \bigwedge^2(\mathbb{C}^4)$, defined in the following way:

$$\phi_\omega(v) = \omega \wedge v.$$

Consider the linear map

$$\begin{aligned} \eta : \bigwedge^{2}(\mathbb{C}^4) &\to \hom(\mathbb{C}^4, \bigwedge^{3}(\mathbb{C}^4)) \\ : \qquad \omega &\longmapsto \phi_\omega. \end{aligned}$$

If ω in $\bigwedge^2(\mathbb{C}^4)$ is totally decomposable, then ϕ_ω has rank $n-k$. Thus, $G(2,4)$ is defined as the vanishing of the $(n-k+1)\times(n-k+1)$ minors of the matrix ϕ_ω, and it is a projective subvariety of $\mathbb{C}P^n$. Hence, it is Kähler by the restriction of the Fubini-Study metric.

Let's see if $G(2,4)$ may be Kähler -Einstein. As above, we can describe $G(2,4)$ as $M_0(2,4)/GL_2$, where $M_0(2,4)$ denotes the space of 2×4 matrices of full rank. Then the homogeneous coordinates of $G(2,4)$ in $\mathbb{P}^5$ are given by the determinants of the 2×2 minors of the matrix by the Plücker embedding. Recall from the first example that the metric on $\mathbb{P}^5$ is given by

$$\omega_{FS} = \partial\bar{\partial}\log(\|z\|^2).$$

Thus, when restricted to $G(2,4)$, the metric on $G(2,4)$ when represented as $M_0(2,4)/GL_2$ is given by

$$\|A\| = \partial\bar{\partial}\log\left(\sum_{\substack{\tilde{A}\\ 2\times 2 \text{ minors}}} \|\det(\tilde{A})\|^2\right), \qquad \text{for } A\in M_0(2,4).$$

Note that this is well defined since if $B\in GL_2$, then

$$\begin{aligned} \|BA\| &= \partial\bar{\partial}\log\left(\sum_{\substack{\tilde{A}\\ 2\times 2 \text{ minors}}} \|\det(B\tilde{A})\|^2\right) \\ &= \partial\bar{\partial}\log\left(\sum_{\substack{\tilde{A}\\ 2\times 2 \text{ minors}}} \|\det(B)\|^2\|\det(\tilde{A})\|^2\right) \\ &= \partial\bar{\partial}\log\left(\sum_{\substack{\tilde{A}\\ 2\times 2 \text{ minors}}} \|\det(\tilde{A})\|^2\right) + \partial\bar{\partial}\log(\|\det B\|^2) \\ &= \partial\bar{\partial}\log\left(\sum_{\substack{\tilde{A}\\ 2\times 2 \text{ minors}}} \det(\|(\tilde{A})\|^2\right) \\ &= \|A\| \end{aligned}$$

Locally, every plane $L \in G(2,4)$ can be described by a matrix

$$\begin{bmatrix} 1 & 0 & a_{11} & a_{12} \\ 0 & 1 & a_{21} & a_{12} \end{bmatrix}$$

by switching some columns if necessary. Then the homogeneous coordinates are given by $[1 : a_{21} : a_{22} : a_{11} : a_{12} : \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}]$. Note that this is a hypersurface in $\mathbb{P}^5$. In fact, it's a quadric hypersurface given by the vanishing of $F(z) = z_0 z_5 - z_2 z_3 + z_1 z_4$. The relations that result by taking the determinants of the $k \times k$ minors of a matrix of the above form are known as the **Plücker relations**.

From above, we see then that $G(2,4)$ has a positive canonical class, i.e. $(n + 1 - d)\omega_{FS} \in c_1(M)$. In this case, we have that $4\,\omega_{FS}|_{G(2,4)} \in c_1(M)$. It's not clear from this, however, that a Kähler -Einstein metric exists on $G(2,4)$. As it turns out, the Fubini-Study metric on $\mathbb{C}P^5$ when restricted to the image of $G(2,4)$ is in fact Kähler -Einstein.

We have

$$\sum_{i=0}^{n} \|\frac{\partial F}{\partial z_i}\|^2 = \|z\|^2 = \|z_0\|^2(1 + \|w\|^2).$$

From the equation (0.2), we have for $G(2,4)$ a quadric hypersurface in $\mathbb{C}P^5$,

$$\begin{aligned} Ric(\omega_{FS}|_{G(2,4)}) &= -\partial\bar{\partial} \log \frac{\|z_0\|^2((1 + \|w\|^2)}{(1 + \|w\|^2)^5} \\ &= 4\partial\bar{\partial} \log(1 + \|w\|^2) \\ &= 4\omega_{FS}|_{G(2,4)}. \end{aligned}$$

Hence, $G(2,4)$ is an example of a Kähler -Einstein manifold with $c_1(M) > 0$.

Example 5 (A non-Kähler complex manifold)**.**

Just as the first rather simple example of $\mathbb{C}^n$ was necessary to exhibit that our basic manifolds are in fact Kähler, this example demonstrates that not all complex manifolds are Kähler .

Consider the 3-sphere, S^3 defined by $\{z = (z_1, z_2) \in \mathbb{C}^2 | \|z_1\|^2 + \|z_2\|^2 = 1$. Let

$$f : S^3 \times \mathbb{R} \xrightarrow{\cong} \mathbb{C}^1 \backslash \{0\}$$

be the diffeomorphism given by

$$f(z_1, z_2, t) = (e^t z_1, e^t z_2).$$

Conceptually, the map f is shrinking and expanding S^3 in $\mathbb{C}^2$ exponentially by the parameter t. $\mathbb{Z}$ acts on $S^3 \times \mathbb{R}$ in the following way: for $m \in \mathbb{Z}$,

$$m \cdot (z_1, z_2, t) \longmapsto (z_1, z_2, t + m).$$

It's clear that the quotient space under this action is $S^3 \times S^1$. Via f, the action of $\mathbb{Z}$ on $S^3 \times \mathbb{R}$ gives us an action of $\mathbb{Z}$ on $\mathbb{C}^2\backslash\{0\}$, defined by

$$m \cdot (z_1, z_2) = (e^m z_1, e^m z_2).$$

For fixed m, the mapping above is clearly and automorphism of $\mathbb{C}^2\backslash\{0\}$ and it's easy to see that the action of $\mathbb{Z}$ on $\mathbb{C}^2\backslash\{0\}$ gives us a free action of a properly discontinuous subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{C}^2\backslash\{0\})$. This gives us the following commutative diagram.

$$\begin{tikzcd}
S^3 \times \mathbb{R} \arrow[r, "f"] \arrow[d] & \mathbb{C}^2\backslash\{0\} \arrow[d] \\
(S^3 \times \mathbb{R})/\mathbb{Z} \arrow[r, "\tilde{f}"] \arrow[d, equal, "\wr"] & (\mathbb{C}^2\backslash\{0\})/\Gamma \arrow[d, equal, "\wr"] \\
S^3 \times S^1 & X
\end{tikzcd}$$

Since X is the quotient space of a complex vector space by the free action of a discrete, properly discontinuous group of automorphisms, X is a complex manifold. Via $\tilde{f}$, X is diffemorphic to $S^3 \times S^1$, and so it has the same betti numbers as $S^3 \times S^1$. In particular, the first betti number for $S^3 \times S^1$ is 1. By the Hodge decomposition on Kähler manifolds, the betti numbers of odd degree must be even on Kähler manifolds. Hence, X is a complex manifold which is not Kähler .

Remark. A result of Kodaira asserts that any compact complex manifold homeomorphic to $S^3 \times S^1$ is of the form $\mathbb{C}^2\backslash\{0\}/\Gamma$ for some appropriate properly discontinuous, discrete subgroup Γ of $\mathrm{Aut}(\mathbb{C}^2\backslash\{0\})$. Such manifolds are called **Hopf surfaces**.

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