LIE ALGEBRA COHOMOLOGY

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ABSTRACT. These are notes used for a talk given on Lie algebra cohomology. We follow section 2 of Bott's *Homogeneous Vector Bundles.* The definitions are done abstractly, and for general nice lie algebras, but being geometrically minded, I've placed in parentheses "secretly" how I think of these objects.

1. DEFINITION

Let $\mathfrak{g}$ be a Lie algebra, and F be a $\mathfrak{g}$-module. Let $A^n\mathfrak{g}, F$ be the space of n-alternating maps from $\mathfrak{g}$ to F. Let $A(\mathfrak{g}, F) = \sum_n A^n(\mathfrak{g}, F)$.

Secretly, F is the set of C^∞ functions on some smooth manifold, X. $\mathfrak{g}$ consists of all the derivations of F. Thus, an alternating map from $\mathfrak{g}$ to F is a differential form on X.

We have a $\mathfrak{g}$-module structure on $A(\mathfrak{g}, F)$ defined by the operator θ. That is, for $x, z_i \in \mathfrak{g}$, $n > 0$, $w \in A^n(\mathfrak{g}, F), n > 0$,

$$\theta(x) \cdot w(z_1, z_2, \ldots, z_n) = x \cdot w(z_1, \ldots, z_n) - \sum_i w(z_1, \ldots, [x, z_i], \ldots, z_n).$$

For $n = 0$, we just have the usual action of $\mathfrak{g}$ on F.

Secretly, θ is the Lie derivative, i.e. $\theta(X) = \mathfrak{L}_X$.

We also have the operator $i(x)$ for $x \in \mathfrak{g}, n > 0$, given by

$$(i(x)w)(y_1, \ldots y_{n-1}) = w(x, y_1, \ldots, y_{n-1}).$$

Secretly, this is interior multiplication/contraction

For $n = 0$, $i(x) = 0$.

One can check that for $x, y \in \mathfrak{g}$,

$$i(x)\theta(y) - \theta(y)i(x) = i([x, y]).$$

Secretly, just Cartan's magic formula

These two operators uniquely determine a boundary operator, d by the requirement:

$$i(x)d + di(x) = \theta(x) \qquad \text{for all } x \in \mathfrak{g} \tag{1.1}$$

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We can also write down the explicit formula for d:

(1.2)

$$dw(x_1, \dots, x_{n+1}) = \sum_i (-1)^{i+1} x_i \cdot w(x_1, \dots, \hat{x}_i,]dots, x_{n+1})$$
$$+ \sum_{p<q} (-1)^{p=q} w([x_p, x_q], x_1, \dots, \hat{x}_p, \dots, \hat{x}_q, \dots, x_{n+1}).$$

By induction, d^2 is identically zero. Let $w \in A^0(\mathfrak{g}, F)$. Then $i(x)dw = \theta(x)w$, by (1.1) and $i(x)w = 0$ on $A^0(\mathfrak{g}, F)$. Now,

$$(i(x_1)d^2w)(x_2) + (di(x)dw)x_2) = d(\theta(x_1)w)(x_2) \tag{1.3}$$

$$d^2w(x_1, x_2) + d(dw(x_1))(x_2) = d(x_1 \cdot w)(x_2) \tag{1.4}$$

$$d^2w(x_1, x_2) = 0 \tag{1.5}$$

Thus, by the inductive hypothesis,

$$d^2w(x_1, x_2, \dots, x_n) = d^2(i(x_1)w(x_2, x_3, \dots, x_n)) = 0.$$

The derived module of $A(\mathfrak{g}, F)$ with respect to d is the cohomology module of $\mathfrak{g}$ with coefficients in F.

Relative cohomology is defined in the following way. If $\mathfrak{u} \subset \mathfrak{g}$ is a subalgebra, let $A(\mathfrak{g}, \mathfrak{u}, F) \subset A(\mathfrak{g}, F)$ be the subspace of alternating linear maps that are annihilated by $i(x)$ and $\theta(x)$ for $x \in \mathfrak{u}$. It is easy to check by the previously stated relations between the operators that this subspace is invariant under d, and hence a subcomplex of $A(\mathfrak{g}, F)$. This is out relative Lie algebra cohomology, which we will denote as $H^*(\mathfrak{g}, \mathfrak{u}, F)$.

Just as with spaces and singular cohomology, homomorphisms between Lie algebras induce homomorphisms between their cohomology groups. In fact, we can do this a little more generally and discuss induced homomorphisms with homomorphims between coefficients.

Let $\lambda : \mathfrak{g}' \to \mathfrak{g}$ be a Lie homomorphism, and $\eta : F \to F'$ be a linear map of modules. Then we define the map $\lambda^* \otimes \eta w : A(\mathfrak{g}, F) \to A(\mathfrak{g}', F')$ by

$$(\lambda^* \otimes \eta w)(x_1, \dots, x_n) = \eta w(\lambda x_1, \dots, \lambda x_n),$$

for $x_i \in \mathfrak{g}', w \in A^n(\mathfrak{g}, F)$. The maps λ and η are compatible if $\eta(\lambda x \cdot f) = x \cdot \eta f)$ for $x \in \mathfrak{g}', f \in F$. For compatible maps, $\lambda^* \otimes \eta$ commutes with the differential operators of the complexes and thus is a chain map, and so induces a homomorphism of cohomology groups. Similarly, if $\mathfrak{u} \subset \mathfrak{g}$ and $\mathfrak{u}' \subset \mathfrak{g}'$ are subalgebras with $\lambda(\mathfrak{u}') \subset \mathfrak{u}$, then $\lambda^* : A(\mathfrak{g}, \mathfrak{u}, F) \to A(\mathfrak{g}', \mathfrak{u}', F)$ commutes with d and we have an induced homomorphism of relative Lie algebra cohomology groups.

We continue a bit further with this abstract nonsense.

Let Λ be a commutative ring over $\mathbb{C}$. Let $\mathfrak{D}$ be all the derivations of Λ. Then $\mathfrak{D}$ is a Lie algebra. If F is a Λ-module, then we define $B_\Lambda(\mathfrak{D}, F)$ to be the space of all alternating, Λ-linear maps. To talk about the cohomology of $\mathfrak{D}$ with coefficients in F, we need F to be a $\mathfrak{D}$ module. We say that F has compatible $\mathfrak{D}$-module and Λ module structures if the following holds:

$$x \cdot (\lambda f) = (x \cdot \lambda) f + \lambda (x \cdot f) \qquad \text{for } x \in \mathfrak{g}, f \in F, \lambda \in \Lambda \tag{1.6}$$

That is, since F is always it's own module, we want $\mathcal{D}$ to satisfy the product rule.

Secretly, when F is the set of smooth sections of a vector bundle, the compatiblity condition is just the obstruction to a flat connection.

If the compatibility condition is satisfied, then we can identify $B_\Lambda(\mathfrak{D}, F)$ as a subspace of the $A(\mathfrak{D}, F)$. Let $A_\Lambda(\mathfrak{D}, F)$ denote the subspace of $B_\Lambda(\mathfrak{D}, F)$ that is stable under $\theta(x)$. This is a subcomplex stable under $\theta(x), i(x)$, and d. We denote the derived cohomology modules of this subcomplex by $H_\Lambda(\mathfrak{D}, F)$.

Example 1.1. Given a smooth manifold X, let $\mathfrak{D}$ be all the derivations of $\Lambda = F = C^\infty-$complex valued functions on X. Then $\mathfrak{D}$ certainly satisfies (1.6) and the derived cohomology module $H^*_\Lambda(\mathfrak{D}, F)$ is the De Rham comohology $H^*_{DR}(X, \mathbb{C})$ of X.

Example 1.2. Let $E \to X$ be a complex analytic vector bundle over a complex analytic manifold X. Let

$$\begin{aligned} F &= \Gamma_\infty(E) \qquad \text{(smooth sections of } E) \\ \Lambda &= C^\infty\text{-complex valued functions on} X \end{aligned}$$

Then $B_\Lambda(\mathfrak{D}, F)$ is the space of E- valued forms on X.

(For $\omega \in B_\Lambda(\mathfrak{D}, F), \omega(X_1, X_2, \ldots, X_n) = f$ where f is a section of E. At a point p, ω eats vectors and produces an element in E_p.)

Unfortunately, in general, we can't define a compatible $\mathfrak{D}$ structure on F since we only have a flat connection when our E is locally constant.

However, $\mathfrak{D}$ splits in the following way.

$$\mathfrak{D} = \mathfrak{D}_\alpha + \mathfrak{D}_\beta$$

where $\mathfrak{D}_\alpha$ consisting of the derivations that annihilate the antiholomorphic functions (*Secretly,* $\frac{\partial}{\partial z^i}$), and $\mathfrak{D}_\beta$ the derivations that annihilate the holomorphic functions (*Secretly,* $\frac{\partial}{\partial \bar{z}^i}$).

Since $E \to X$ is a complex analytic vector bundle, the transition functions are holomorphic, so we can define a compatible $\mathfrak{D}_\beta$-connection in the following way. If $s \in \Gamma_\infty(E)$, and $\{e_1, e_2, \ldots, e_n\}$ are a basis for E, then locally

$$s = \sum_i \lambda_i e_i \qquad \text{where } \lambda_i \text{ are holomorphic functions}$$

Then,

$$\bar{\partial} s = \sum_i (\bar{\partial} \lambda_i) e_i$$

is well defined since other representations of s differ by a holomorphic transition function, that is annihilated by elements in $\mathfrak{D}_\beta$. Thus, this extends to a global definition.

The $\Lambda-$module $A_\Lambda(\mathfrak{D}_\beta, F)$ is the complex of E-valued forms on X of type $(0, q)$, and

$$H^q(\mathfrak{D}_\beta, F) = H^{0,\,q}(X; E).$$

How do we get at the $H^{p,\,q}(X; E)$?

Consider the modules, $B^p_\Lambda(\mathfrak{D}_\alpha, F)$. Recall, these modules consist of Λ-linear alternating maps of degree p from $\mathfrak{D}_\alpha$ to F. These modules admit a compatible $\mathfrak{D}_\beta$ structure in the following way.

$$(x \cdot w)(z_1, \ldots, z_p) = x \cdot w(z_1, \ldots, z_p) - \sum_i w(x_i, \ldots, \alpha([x, z_i]), \ldots, z_p)$$

where α is the projection map from $\mathfrak{D}$ to $\mathfrak{D}_\alpha$, $x, z_i \in \mathfrak{D}_\beta$, and $w \in B^p_\Lambda(\mathfrak{D}_\alpha, F)$.

Then, $A_\Lambda(\mathfrak{D}_\beta, B^p_\Lambda(\mathfrak{D}_\alpha, F))$ is a complex, and

$$H^q_\Lambda(\mathfrak{D}_\beta, B^p_\Lambda(\mathfrak{D}_\alpha, F)) = H^{p,\,q}(X; E).$$

This is the (p, q) cohomology of E.

Example 1.3. Let G be a Lie group, and let Λ = complex-valued C^∞ functions on G. We have an injection

$$\rho : \mathfrak{g} \to \mathfrak{D}$$

. Note that this is an injection since $\mathfrak{g}$ consists of all left invariant vector fields on G, while $\mathfrak{D}$ consists of all vector fields on G.

This extends to a $\Lambda-$ linear homomophism

$$\rho : \Lambda \otimes \mathfrak{g} \to \mathfrak{D}.$$

This is a bijective since $\mathfrak{g}$ is a $\Lambda-$ basis for $\mathfrak{D}$. That is, every vector fields can be described as the finite sum of smooth functions on G times the basis elements of the tangent field at the identity. Hence, we have an induced map

$$\rho^* : A_\Lambda(\mathfrak{D}, \Lambda) \to A(\mathfrak{g}, \Lambda),$$

which is bijective and a chain homomorphism. Thus,

$$H^*_\Lambda(\mathfrak{D}, \Lambda) = H^*(\mathfrak{g}, \Lambda),$$

or rather,

$$H^*(G, \mathbb{C}) = H^*(\mathfrak{g}, \Lambda).$$

Thus, the cohomology of a Lie group is the cohomology of its lie algebra.

Example 1.4. We generalize the above. If U is a connected, closed subgroup of G, and $X = G/U$, i.e. X is a homogeneous space, then the projection map $p : G \to G/U$ induces a map p^* that embeds the chain complex of differential forms on G/U into the forms on G. The image of this map consists of the forms which are invariant under right translations by U, and annihilate vector fields parallel to U. Thus, p^* maps into $A(\mathfrak{g}, \mathfrak{u}, \Lambda)$. Since U is connected, p^* is onto, and we have that

$$H^*(X, \mathbb{C}) = H^*(G, U, \mathbb{C}) = H^*(\mathfrak{g}, \mathfrak{u}, \Lambda).$$

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