

L^2 NORMS OF SECTIONS OF $\mathcal{O}(d)$ ON $\mathbb{C}P^n$

TSECHING LIEN

ABSTRACT. We compute the L^2 norms of sections of $\mathcal{O}(d)$ over $\mathbb{C}P^n$, with respect to the Fubini-Study metric.

Let $\mathcal{O}(-1)$ denote the tautological line bundle on $\mathbb{C}P^n$. Formally,

$$\mathcal{O}(-1) = \{([z], \lambda z) : \lambda \in \mathbb{C}\}.$$

That is, we can think of a fiber of $\mathcal{O}(-1)$ lying over $[z]$ to be the line in $\mathbb{C}P^n$ represented by $[z]$. A metric on $\mathcal{O}(-1)$ is naturally given by the standard hermitian metric in \mathbb{C}^n . Since $\mathcal{O}(1)$ is the dual of $\mathcal{O}(-1)$, the metric on $\mathcal{O}(-1)$ induces a dual metric on $\mathcal{O}(1)$ in the following way. At a point $[z]$, given an element $([z], e)$ of $\mathcal{O}(-1)$, we can define $([z], \delta_e)$ to be the dual element that takes the value 1 on e and 0 on the orthogonal complement. Then the metric on $\mathcal{O}(-1)$ induces a metric on $\mathcal{O}(1)$ given by

$$\|\delta_e\|^2 = \frac{1}{\|e\|^2}.$$

On $\mathcal{O}(-1)$, over the open set $U_0 = \{[z] \in \mathbb{C}P^n : z_0 \neq 0\}$, pick our local section e to be $e([z]) = [1, \delta_e[1, w_1, \dots, w_n] \in \mathbb{C}^n$. Then the dual to that section δ_e is given by the homogeneous degree 1 polynomial z_0 , since $z_0(e) = 1$. Thus, z_0 is a local section to $\mathcal{O}(1)$ over U_0 .

$\mathcal{O}(d)$ is the d^{th} tensor product of $\mathcal{O}(1)$. Thus, a basis element for $\mathcal{O}(d)$ over U_0 is given by z_0^d . Now, sections f of $\mathcal{O}(d)$ are given by the homogeneous polynomials of degree d . Over U_0 , we have that $f = f_0 \cdot \delta_{e_0}$, where $\delta_{e_0} = z_0^d$, and $f_0 = f(1, w_1, \dots, w_n)$. Hence, $\|f\|_{U_0}^2 = \|\tilde{f}\|^2 \|\delta_{e_0}\|^2$.

Now,

$$\begin{aligned} \|\delta_{e_0}\|^2 &= \frac{1}{(1 + \|w_1\|^2 + \dots + \|w_n\|^2)^d} \\ &= \frac{(\|z\|^2)^d}{(\|z_0\|^2 + \dots + \|z_n\|^2)^d}. \end{aligned}$$

This gives us, over U_0 for a section f of $\mathcal{O}(d)$

$$\|f\|^2 = \frac{\|f_0\|^2}{(1 + \|w_1\|^2 + \dots + \|w_n\|^2)^d}.$$

Globally, this is given by

$$\|f\|^2 = \frac{\|f(z_0, z_1, \dots, z_n)\|^2}{(\|z_0\|^2 + \dots + \|z_n\|^2)^d}.$$

We can check that the above equation is well defined over $\mathbb{C}P^n$ since f is homogeneous of degree d as is the denominator.

We're interested in the L^2 norm of these sections with respect to ω_{FS} , the Fubini-Study metric. The L^2 norm of f is defined to be

$$\|f\|_{L^2}^2 = \int_{\mathbb{C}P^n} \frac{\|f(z_0, z_1, \dots, z_n)\|^2}{(\|z_0\|^2 + \dots + \|z_n\|^2)^d} \omega_{FS}^n$$

We compute this locally, over the open set U_0 . Note that it is sufficient to just do the computation when f is a monomial. Suppose $f = z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$, where $i_0 + \dots + i_n = d$. Then we have

$$\begin{aligned} & \int_{U_0} \frac{\|f(z_0, z_1, \dots, z_n)\|^2}{(\|z_0\|^2 + \dots + \|z_n\|^2)^d} \omega_{FS}^n \\ &= \int_{\mathbb{C}^n} \frac{\|f_0(1, w_1, \dots, w_n)\|^2}{(1 + \|w\|^2)^d} \omega_{FS}^n \\ &= \int_{\mathbb{C}^n} \frac{\|w_1\|^{2i_1} \|w_2\|^{2i_2} \dots \|w_n\|^{2i_n}}{(1 + \|w\|^2)^d} \omega_{FS}^n. \end{aligned}$$

By a computation (or one can reference the other note "Examples of Kähler Manifolds,"), on U_0 in $\mathbb{C}P^n$,

$$\omega_{FS}^n = \frac{1}{(1 + \|w\|^2)^{n+1}} dw_1 \wedge \dots \wedge dw_n.$$

Let $w = r\tilde{w}$ where $\tilde{w} = \frac{w}{\|w\|}$, and let $d\sigma$ denote the surface measure for a sphere.

$$\begin{aligned} & \int_{\mathbb{C}^n} \frac{\|w_1\|^{2i_1} \|w_2\|^{2i_2} \dots \|w_n\|^{2i_n}}{(1 + \|w\|^2)^d} \omega_{FS}^n \\ &= \int_{\mathbb{C}^n} \frac{\|w_1\|^{2i_1} \|w_2\|^{2i_2} \dots \|w_n\|^{2i_n}}{(1 + \|w\|^2)^d} \frac{1}{(1 + \|w\|^2)^{n+1}} dw_1 \wedge \dots \wedge dw_n \\ &= \int_{r=0}^{\infty} \int_{S^{2n-1}} \frac{f(r\tilde{w})}{(1 + r^2)^d} \frac{r^{2n-1}}{(1 + r^2)^{n+1}} dr d\sigma \\ &= \int_{r=0}^{\infty} \int_{S^{2n-1}} \frac{r^{2(i_1 + \dots + i_n)} f(\tilde{w})}{(1 + r^2)^{d+n+1}} r^{2n-1} dr d\sigma \\ &= \int_{r=0}^{\infty} \int_{S^{2n-1}} \frac{r^{2(i_1 + \dots + i_n + n) - 1}}{(1 + r^2)^{d+n+1}} f(\tilde{w}) dr d\sigma \\ &= \left(\int_{r=0}^{\infty} \frac{r^{2(i_1 + \dots + i_n + n) - 1}}{(1 + r^2)^{d+n+1}} dr \right) \left(\int_{S^{2n-1}} f(\tilde{w}) d\sigma \right) \end{aligned}$$

So we need only to compute the two integrals. We start with the radial integral.

0.1. Radial integral. Define for $m, n \in \mathbb{Z}$ such that $m > n$, the integral

$$I(m, n) = \int_0^\infty \frac{x^{2m-1}}{(1+x^2)^n} dx.$$

Integration by parts gives us then that

$$\begin{aligned} I(m, n) &= -\frac{r^{2m-2}}{2(n-1)(1+r^2)^{n-1}} \Big|_0^\infty + \frac{m-1}{n-1} I(m-1, n-1) \\ &= \frac{m-1}{n-1} I(m-1, n-1), \end{aligned}$$

since the first term vanishes when $r = 0$ and at infinity, $-\frac{r^{2m-2}}{2(n-1)(1+r^2)^{n-1}} = 0$ by L'Hopital's since $2m-2 < 2n-2$.

Thus, we have

$$\begin{aligned} I(m, n) &= \frac{m-1}{n-1} I(m, n) \\ &= \frac{(m-1)!(n-m)!}{(n-1)!} I(1, n-m+1) \\ &= \frac{(m-1)!(n-m)!}{(n-1)!} \int_0^\infty \frac{x}{(1+x^2)^{n-m+1}} dx \\ &= \frac{(m-1)!(n-m)!}{(n-1)!} \frac{1}{2(n-m)(1+x^2)^{n-m}} \Big|_0^\infty \\ &= \frac{(m-1)!(n-m)!}{(n-1)!} \frac{1}{2(n-m)} \\ &= \frac{(m-1)!(n-m-1)!}{2(n-1)!} \end{aligned}$$

Thus,

$$\begin{aligned} \int_{r=0}^\infty \frac{r^{2(i_1+\dots+i_n+n)-1}}{(1+r^2)^{d+n+1}} dr &= I(i_1 + \dots + i_n + n, d + n + 1) \\ &= \frac{(i_1 + \dots + i_n + n - 1)!(d - i_1 - \dots - i_n)!}{2(d+n)!} \\ &= \frac{(i_1 + \dots + i_n + n - 1)!(i_0)!}{2(d+n)!} \end{aligned}$$

0.2. Integrating a monomial over a sphere. Recall that the Gamma function is defined as

$$\Gamma(t) = \int_0^\infty x^{t-1} \exp^{-x} dx.$$

For $n \in \mathbb{Z}$, $\Gamma(n) = (n-1)!$. With an appropriate change of variables, an equivalent definition of $\Gamma(t)$ is

$$\Gamma(t) = 2 \int_0^\infty x^{2t-1} e^{-x^2} dx.$$

Denote the usual Lebesgue measure on \mathbb{C}^n by dm

Given a monomial $P(w) = \|w_1\|^{2i_1} \dots \|w_n\|^{2i_n}$ defined over \mathbb{C}^n , we have

$$\begin{aligned} \int_{\mathbb{C}^n} P(w) e^{-\|w\|^2} dm &= \prod_{k=1}^n \int_{\mathbb{C}} \|w_k\|^{2i_k} e^{-\|w_k\|^2} dm \\ &= \prod_{k=1}^n \int_{r=0}^\infty \int_{\theta=0}^{2\pi} r^{2i_k} e^{-r^2} r dr d\theta \\ &= \prod_{k=1}^n 2\pi \int_{r=0}^\infty r^{2(i_k+1)-1} e^{-r^2} dr \\ &= \pi^n \prod_{k=1}^n \Gamma(i_k + 1) \end{aligned}$$

On the other hand, if we make an immediate switch into polar coordinates, then we have

$$\begin{aligned} \int_{\mathbb{C}^n} P(w) e^{-\|w\|^2} dm &= \int_{r=0}^\infty \int_{S^{2n-1}} P(r\tilde{w}) e^{-r^2} r^{2n-1} dr d\sigma \\ &= \left(\int_{r=0}^\infty r^{2(i_1+\dots+i_n+n)-1} e^{-r^2} dr \right) \left(\int_{S^{2n-1}} P(\tilde{w}) d\sigma \right) \\ &= \frac{1}{2} \Gamma(i_1 + \dots + i_n + n) \left(\int_{S^{2n-1}} P(\tilde{w}) d\sigma \right) \end{aligned}$$

Dividing through then, we have

$$\begin{aligned} \left(\int_{S^{2n-1}} P(\tilde{w}) d\sigma \right) &= \frac{2\pi^n \prod_{k=1}^n \Gamma(i_k + 1)}{\Gamma(i_1 + \dots + i_n + n)} \\ &= \frac{\pi^n i_1! i_2! \dots i_n!}{2(i_1 + \dots + i_n + n - 1)!} \end{aligned}$$

Hence, we have that for a homogeneous polynomial of degree d over U_0 in $\mathbb{C}P^n$,

$$\begin{aligned} \|f\|_{L^2}^2 &= \left(\int_{r=0}^{\infty} \frac{r^{2(i_1+\dots+i_n+n)-1}}{(1+r^2)^{d+n+1}} dr \right) \left(\int_{S^{2n-1}} f(\tilde{w}) d\sigma \right) \\ &= \left(\frac{(i_1 + \dots + i_n + n - 1)!(i_0)!}{2(d+n)!} \right) \left(\frac{2\pi^n i_1! i_2! \dots i_n!}{(i_1 + \dots + i_n + n - 1)!} \right) \\ &= \pi^n \frac{i_0! i_1! \dots i_n!}{(d+n)!} \end{aligned}$$

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, 480 LINCOLN DRIVE,
MADISON, WI 53706

E-mail address: lien@math.wisc.edu