# NOTES ON THE SOBOLEV CONSTANTS AND RELATED ISOPERIMETRIC INEQUALITIES

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Abstract. Here's what I understand, or don't understand about Sobolev spaces on a Riemannian manifold.

Given a Riemannian $m$-manifold $M$, we want to know under what conditions the Sobolev embeddings are valid for our manifold. Recall that if our manifold is $\mathbb{R}^m$, then the Sobolev embeddings are valid.

**Theorem 0.1** (Sobolev Embedding for $\mathbb{R}^m$)**.** *Let $q \in [1, m)$ and let $p$ be such that $1/p = 1/q - 1/m$. For any $\varphi \in H_1^q(\mathbb{R}^m)$,*

$$\left(\int_{\mathbb{R}^m} |\varphi|^{m/(m-1)} dx\right)^{(m-1)/m} \leq \frac{p(m-1)}{2m} \left(\int_{\mathbb{R}^m} |\nabla\varphi| dx\right)^{1/q}. \tag{0.1}$$

*In particular, for any real numbers $1 \leq q < p$, and any integers $0 \leq n < k$ satisfying $1/p = 1/q - (k-n)/m$, we have that $H_k^q(\mathbb{R}^m) \subset H_n^p(\mathbb{R}^m)$.*

Let's consider embedding $H_1^q(\mathbb{R}^m \subset H_0^p(\mathbb{R}^m) = L^p(\mathbb{R}^m)$ for now, i.e., we want to show that

$$\left(\int_{\mathbb{R}^n} |\varphi|^p dx\right)^{1/p} \leq \frac{p(m-1)}{2m} \left(\int_{\mathbb{R}^n} |\nabla\varphi| dx\right).$$

First, we show that

**Lemma 0.2.**

$$\left(\int_{\mathbb{R}^m} |\varphi|^{m-1/m} dx|\right)^{m-1/m} \leq \prod_{i=1}^{m} \left(\int_{\mathbb{R}^m} |\frac{\partial\varphi}{\partial x^i}| dx\right)^{1/n}$$

*Proof.* We do the proof for $\mathbb{R}^3$, but the argument is the same for $\mathbb{R}^m$. Let $(x_0, y_0, z_0)$ be the coordinates of a point, $P \in \mathbb{R}^3$, and $D_x, D_y, D_z$ be lines through $P$ that are parallel the the $x, y$, and $z$ axes, respectively. So,

$$\varphi(P) = \int_{-\infty}^{x_0} (\partial_x\varphi)(x, y_0, z_0) dx = -\int_{x_0}^{\infty} (\partial_x\varphi)(x, y_0, z_0) dx.$$

As a consequence,

$$|\varphi(P)| \leq \frac{1}{2} \left(\int_{D_x} |(\partial_x\varphi)(x, y_0, z_0)| dx\right)^{1/2}.$$

Thus,

$$\begin{aligned}|\varphi(P)|^{3/2} \quad \leq \quad & \left(\frac{1}{2}\right)^{3/2}\left(\int_{D_x}|(\partial_x\varphi)(x,y_0,z_0)|dx\right)^{1/2} \\ & \times\left(\int_{D_y}|(\partial_y\varphi)(x_0,y,z_0)|dy\right)^{1/2}\left(\int_{D_z}|(\partial_z\varphi)(x_0,y_0,z)|dz\right)^{1/2}\end{aligned}$$

Integrating over $x$, we have

$$\begin{aligned}\left(\int_{D_x}|\varphi(x,y_0,z_0)|dx\right) \quad \leq \quad & \left(\frac{1}{2}\right)^{3/2}\left(\int_{D_x}|(\partial_x\varphi)(x,y_0,z_0)|dx\right)^{1/2} \\ & \times\left(\int_{D_{xy}}|(\partial_y\varphi)(x,y,z_0)|dxdy\right)^{1/2} \\ & \times\left(\int_{D_{xz}}|(\partial_z\varphi)(x,y_0,z)|dxdz\right)^{1/2},\end{aligned}$$

where $D_xy$ is the plane containing $D_x$ and $D_y$. Then, integrating over $y$ and $z$ gives us

$$\left(\int_{\mathbb{R}^m}|\varphi|^{m/(m-1)}dx\right)^{\frac{m-1}{m}} \leq \frac{1}{2}\prod_{i=1}^{m}\left(\int_{\mathbb{R}^m}|\frac{\partial\varphi}{\partial x^i}|dx\right)^{1/m} \tag{0.2}$$

So we have, for $\varphi \in H_1^1(\mathbb{R}^m)$,

$$\left(\int_{\mathbb{R}^m}|\varphi|^{m/(m-1)}dx\right)^{(m-1)/m} \leq \frac{1}{2}\int_{\mathbb{R}^m}|\nabla\varphi|dx.$$

That is, $H_1^1(\mathbb{R}^m) \subset L^{m/(m-1)}(M)$. Then, by the following lemma, we have $H_k^q(\mathbb{R}^m) \subset H_n^p(\mathbb{R}^m)$. In particular, we have,

$$\left(\int_{\mathbb{R}^m}|\varphi|^p dx\right) \leq \frac{p}{m-1}2m\left(\int_{\mathbb{R}^m}|\nabla\varphi|^q dx\right)^{1/q}.$$

☐

**Lemma 0.3.** *Let $(M,g)$ be a complete Riemannian $m$-manifold. Suppose the embedding $H_1^1(M) \subset L^{m/(m-1)}$ is valid. Then, for any real numbers $1 \leq q < p$ and any integers $0 \leq n < k$ satisfying $1/p = 1/q-(k-n)/m, H_k^q(M) \subset H_n^p(M)$.*

*Proof.* ☐

Okay, so we have the Sobolev embeddings for $\mathbb{R}^m$. Now, for compact manifolds, because we can find a partition of unity subordinate to a finite covering, we have the Sobolev embeddings? The embeddings are compact.

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