

Temperley-Lieb Algebra and the Jones Polynomial
of Families of Knots

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Abstract

Although calculations can often be tedious, the polynomial developed by Vaughan Jones is a useful knot invariant. Jones has proven a formula for any (m, n) torus knot. However, his proof relies on difficult algebraic methods involving von Neumann algebras. We explore an easier, combinatorial method to find the Jones polynomial of a torus knot, looking specifically at the $n = 2$ and $n = 3$ case, and extend our technique to another family of knots known as weaving knots.

Chapter 1

Introduction

Knots are a familiar part of our lives. We learn the square knot probably when we first learned to tie our shoes, and although distinguishing simple knots from one another may not be so difficult, rigorizing this mathematically can be quite a feat. While the ends of knots are often left loose in our everyday lives, when we examine knots in mathematics, however, we join the ends after tying the knot. A knot is defined to be a smooth embedding of S^1 into \mathbb{R}^3 . A trivial knot or an unknot is merely a string with its two ends connected and no crossings.

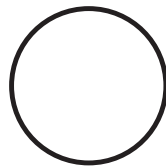


Figure 1.1: Unknot

A nontrivial knot is the trefoil.

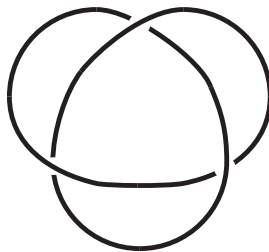


Figure 1.2: Trefoil

Another knot is the figure eight knot.

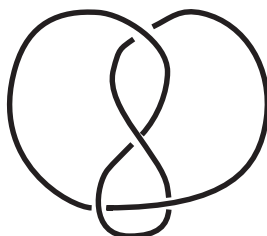


Figure 1.3: Figure eight knot

While knots live in \mathbb{R}^3 , when working with knots, we often project them onto the plane. Because \mathbb{R}^2 sits inside \mathbb{R}^3 in uncountably many ways (consider the set of planes that the z -axis lies in alone!), we would like the choice of which plane we choose to project the knot onto to be not affect the knot itself when we attempt to distinguish it from other knots. Unfortunately, this is not the case. We can end up with many projections of the same knot. For example, the figure eight knot can be projected in at least three different ways.

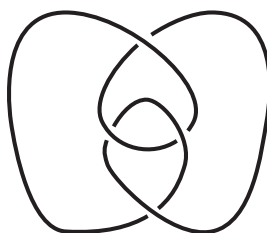


Figure 1.4: Figure eight knot in a different projection

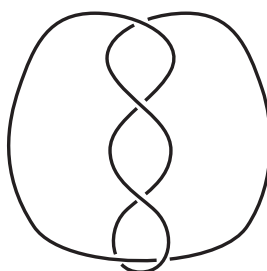


Figure 1.5: Figure eight knot in a third projection

We considered two knots the same if we can transform one to look like the other by rearranging the strands without using scissors and glue. This rearrangement is known as an ambient isotopy. Two projected knots while seemingly different may actually be the same knot if we merely moved the strands around. The three different projections of the figure eight knot are ambient isotopic.

Knots also have applications in other fields such as chemistry. Molecules are formed when protein strands “knot” itself into certain shapes. Knot theory originally grew out of the study of these molecules and finding ways to classify and distinguish them. The main problem in knot theory is being able to tell whether two knots are different. Methods used to distinguish knots are known as knot invariants.

In addition to knots, we have links. Links are created with more than one string, or smooth embeddings of more than one disjoint union of S^1 . For example two unknots would give us a link. The most basic nontrivial link is the Hopf Links.

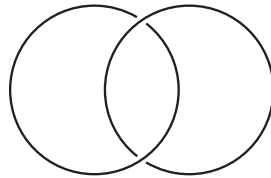


Figure 1.6: Hopf links

Reidemeister moves are ambient isotopic moves that allow us to change one projection of a knot to another. We know two knots are the same if we can get from one knot to the other through a series of Reidemeister moves. A Type I Reidemeister move is uncurling a trivial crossing or creating a trivial crossing.

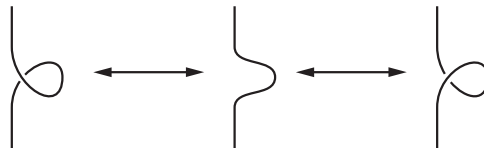


Figure 1.7: Type I Reidemeister move

A Type II Reidemeister move is moving one strand over another strand, or moving one strand under another strand.

Finally, a Type III Reidemeister move is moving a strand past a crossing.

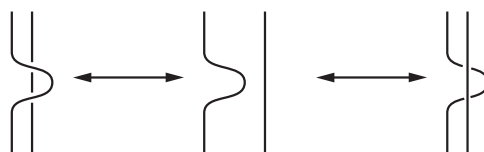


Figure 1.8: Type II Reidmeister Move

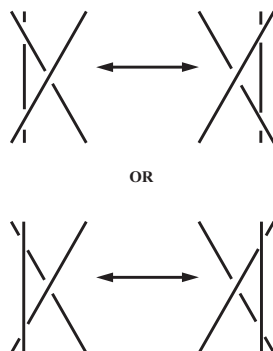


Figure 1.9: Type III Reidmeister Move

A knot invariant needs to be unaffected by Reidemeister moves since two knots are the same if we can transform one knot to another through a series of Reidemeister moves.

The number of crossings in a knot is not a knot invariant for we can take a knot and perform as many Type I Reidemeister moves as we wish, increasing the number of crossings in the projection and still have the end up with the ambient isotopic knot. However, we can speak of the minimal crossing number in a knot. Every knot, except for the unknot, must have a number of crossings. The minimal crossing number is the least number of crossings in all the possible projections of a knot. Two knots that are the same will then have the same minimal crossing number. However, this is not a particularly useful knot invariant for we cannot easily determine the minimal crossing number of a knot. How do we know that there is not another projection of the trefoil that has fewer crossings?

We define the overstrand of a crossing in a projected knot to be the strand that lies on top in the crossing.

The trefoil knot is an example of an alternating knot. If we picked a spot on an alternating knot and “walked” along it, the strands would alternate between overcrossings and undercrossings. An overcrossing is where we are “walking” along the overstrand in the crossing, and an undercrossing is

where we are walking along the understrand. The minimal crossing number of the trefoil is 3 [1]. In fact, the trefoil is the most basic nontrivial knot. Any knot with fewer crossings is necessarily the unknot [1]. This is not difficult to prove. If we take a string and only create one crossing and then connect the ends, then we can perform a Type I Reidemeister move to obtain the unknot.

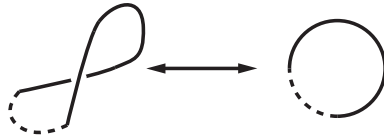


Figure 1.10: A knot with only one crossing is the unknot.

If we create two crossings and then connect the ends, we have four possibilities to consider. Picking one end of the string as our starting point, we either have two overcrossings, two undercrossings, one overcrossing and one undercrossing, or one undercrossing and one overcrossing. With either two overcrossings or two undercrossings, we can perform a Type II Reidemeister move and move one strand out from under the other and obtain the unknot. With an overcrossing and an undercrossing, two Type I Reidemeister moves on each crossing will give us the unknot. Since passing one end of the string through the loop created when creating the first crossing would necessarily give us two additional crossings, we can be certain that the string does not pass through the loop created with the first crossing in creating the second crossing. Hence, we can perform a Type I Reidemeister move on the first crossing and then on the second crossing to obtain the unknot.

A trefoil knot is also a type of torus knot[1]. A torus is a 3 dimensional Euclidean surface that can be best described as the shell of a donut.

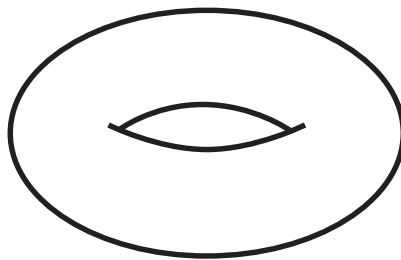


Figure 1.11: Torus

We can think of a torus as taking a sheet of paper and gluing the top

and bottom edges together and then the side edges together. A torus has a meridian and a longitude. The meridian runs along the thickness of the torus, and the longitude runs along the width of the torus (See Figure 1.12).

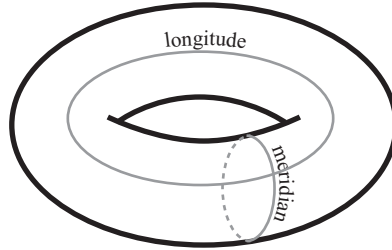


Figure 1.12: The longitude and meridian of the torus

Torus knots lie in the surface of the torus. We describe torus knots with two numbers, presented as (m, n) , where m represents the number of times the string wraps around the torus in the longitudinal direction and n represents the number of times the string wraps around the meridian. The trefoil is a $(3, 2)$ torus knot. Note that an $(m, 1)$ or an $(1, n)$ torus knot is merely the trivial knot. If we merely remove the torus upon creating the knots, we have something that looks similar to a telephone extension cord which can easily be deformed into the unknot. In fact, for a torus knot, m and n are necessarily relatively prime[1]. If they were not relatively prime, then we end up with a number of links rather than knots. The following are some examples of torus knots.

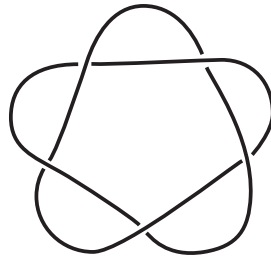


Figure 1.13: $(2,5)$ torus knot

An (m, n) torus knot is the same as an (n, m) torus knot[1]. By performing a series of Reidemeister moves, we can easily transform a (m, n) torus knot into a (n, m) torus knot.

Braids are a set of n strands that are connected to a bar at the top and bottom and each strand heads straight down. The strands can cross over

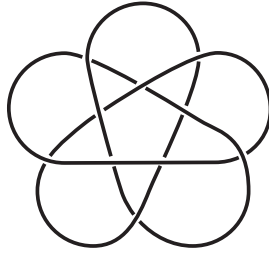


Figure 1.14: (3,5) torus knot

and under one another but it cannot have any strands that double back on itself, creating a loop. For example, a braid might look something like this:



This however, is not a braid because we have a loop that doubles back on itself and does not run straight down.

Every braid can be easily described. Labeling the strands from left to right, we define σ_i as crossing the i^{th} strand over the $i+1^{\text{th}}$ strand. Thus, a σ_3 crossing is where the third strand is crossed over the fourth strand. After each crossing, we re-label the strands from their new positions. After a σ_3 crossing, what was previously the third strand is now the fourth, and the fourth strand is now the third strand. Every σ_i crossing also has an inverse that “undoes” our braid. That is, σ_i^{-1} is where the i^{th} strand crosses under the $i+1$ strand. Its clear then that every braid can be described with a string of σ_i s, and with any string of σ_i s, we can create a braid. The string of σ_i s that describes a particular braid is called the braid word.

If in the braid word we have a σ_i next to a σ_i^{-1} , then as we would



Figure 1.15: Not a braid

when multiplying by an inverse, we could cancel the σ_i , and the σ_i^{-1} since crossing the i^{th} strand over the $i + 1^{\text{th}}$ strand and then after re-labeling, crossing the i^{th} strand under the $i + 1^{\text{th}}$ strand allows us to perform a Type II Reidemeister move.

When we close a braid by attaching the top bar of the braid to the bottom bar of the braid, we have a knot or link with crossings that are described by the braid word. In fact, in 1923, J.W. Alexander proved that every knot or link is the closure of a braid[1]. In particular, we can describe torus knots as the closure of a braid. This is not difficult to imagine. An (m, n) torus knot crosses the meridian m times and the longitude n times. If we cut pick a point on the torus cut along the meridian of the torus, we can deform the torus into a cylindrical tube such that the strands of the knot wraps around the tube. Then we have m distinct strands that runs along the length of the cylindrical tube while encircling its width. Allowing one end of the tube to be the top bar of our braid and the other end of the tube to be the bottom bar of our braid, we can create a braid of m strands whose closure is the (m, n) torus knot.

In fact, the braid word for an (m, n) torus knot is $(\sigma_1, \sigma_2, \dots, \sigma_{m-1})^n$. Since a (m, n) torus knot is the same as an (n, m) torus knot, the closure of a $(\sigma_1, \sigma_2, \dots, \sigma_{m-1})^n$ braid is the same as the closure of a $(\sigma_1, \sigma_2, \dots, \sigma_{n-1})^m$.

Rather than attaching the top bar of a braid to its bottom bar, we can consider “multiplying” braids with the same number of strands by attaching the top bar of one braid to the bottom bar of a second braid.

Figure 1.16: $\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_2\sigma_3\sigma_2^{-1}$ braid

In fact, their braid words multiply nicely as well, since we would just write the first braid word followed by the second braid word. Notice that the braid with no crossings whatsoever acts as an identity element since multiplying any braid with the braid with no crossings gives us back our original braid. Braids and their braid words form an algebraic group with an identity, inverse, and an associative binary operation of multiplication. However, multiplication is not commutative. That is, $a * b \neq b * a$, since the order of the crossings and σ_i 's make a difference[1]. Since each σ_i has an inverse, the braid word for each braid is not unique for we can insert as many σ_i, σ_{i-1} as we wish into a braid word, and perform Type II Reidemeister moves to obtain the same braid. Thus, we want to know when two braid words may represent the same braid. Weve already established that when a σ_i occurs before or after its inverse, the two cancel and we can simplify our braid word. Furthermore, if we have $\sigma_i\sigma_j$ and $|j - i| = 1$, then we can switch the order of σ_i and σ_j in our word. Figure 1.18 demonstrates how we can switch the order of the σ 's.

Also, $\sigma_i\sigma_{i+1}\sigma_i$ is the same as $\sigma_{i+1}\sigma_i\sigma_{i+1}$ since we can perform a Type III Reidemeister move to move the strand from one side of the crossing to the other.

Using the braid representation of a knot, we explore a polynomial knot invariant more in depth in the next chapter.

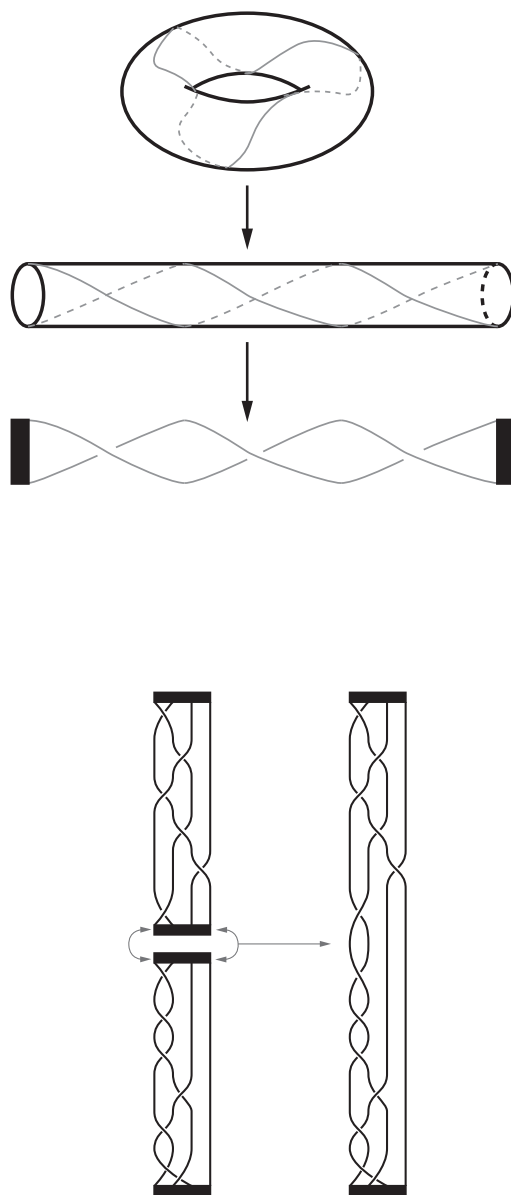


Figure 1.17: Braid conjugation

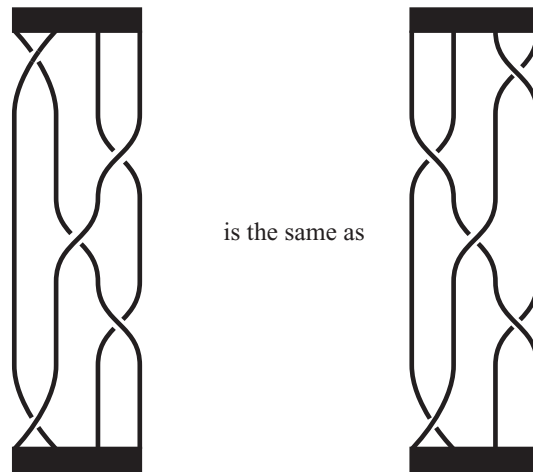


Figure 1.18: Braid word order

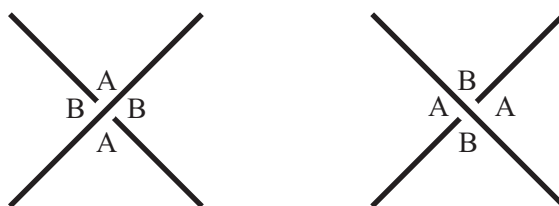
Chapter 2

Bracket/Jones Polynomial and the Temperley-Lieb Algebra

One way to tell knots apart is to assign a knot a polynomial. If two knots have different polynomials, then we can be certain that they are two different knots. This knot invariant is not perfect since two different knots can have the same polynomial. In which case, other knot invariants are necessary for distinguishing these knots. However, because of the ease of calculating the polynomial for a knot, it is a useful knot invariant.

Vaughan Jones developed the Jones polynomial in 1984 as a knot invariant [1]. We explore a method, developed by Louis Kauffman, to calculate the Jones polynomial for any given knot, that is combinatorial in nature and thus, easier to understand[3].

For a crossing in the knot, we rotate the overstrand counterclockwise until it lines up with the understrand. The regions that are swept out during the rotation is labeled A , and the other two regions are labeled B .



We decompose the knot by making two types of splits at the crossing to remove the crossing. An A split is where the two A regions are connected

and a B split is where the B regions are connected. Everytime we make an A split, we multiply the resulting knot by A and each time we make a B split, the resulting knot is multiplied by B . We then continue to decompose each of the resulting knots. The addition of each A split and B split gives us the Kauffman polynomial. We define the Kauffman polynomial of the unknot to be 1. Furthermore, if in our resulting knot we have two unknots, we can remove one by multiplying by a constant, d . That is, if we remove one unknot, we multiply by a constant d ; if we remove two unknots, then we multiply by d^2 .

$$\langle \bigcirc \bigcirc \rangle = d * \langle \bigcirc \rangle$$

$$\langle \bigcirc \bigcirc \bigcirc \rangle = d^2 \langle \bigcirc \rangle$$

We can sum up the method for calculating the Kauffman polynomial by three rules.

Rule 1 $\langle \bigcirc \rangle = 1$

Rule 2 $\langle \times \rangle = A \langle \smile \rangle + B \langle \rangle \langle \rangle$

Rule 3 $\langle \smile \rangle = A \langle \rangle \langle \rangle + B \langle \times \rangle$

As with any other knot invariants, we need to make sure that the polynomial is unaffected by Reidemeister moves. We first consider the Type II Reidemeister move first. We want the polynomial for $\langle \checkmark \rangle$ to be the same as $\langle \rangle \langle \rangle$ when we perform a Type II move.

$$\begin{aligned} \langle \checkmark \rangle &= A \langle \smile \rangle + B \langle \checkmark \rangle \\ &= A(A \langle \smile \rangle + B \langle \checkmark \rangle) + B(A \langle \rangle \langle \rangle + B \langle \smile \rangle) \\ &= A(A \langle \smile \rangle + Bd \langle \smile \rangle) + B(A \langle \rangle \langle \rangle + B \langle \smile \rangle) \\ &= (A^2 + ABd + B^2) \langle \smile \rangle + BA \langle \rangle \langle \rangle \end{aligned} \tag{2.1}$$

Since we want $(A^2 + ABd + B^2) \langle \smile \rangle + BA \langle \rangle \langle \rangle = \langle \rangle \langle \rangle$ so that the polynomial remains unchanged, it is evident that we must make $B = A^{-1}$. Furthermore, we see that in order for $A^2 + ABd + B^2 = A^2 + d + A^{-2} = 0$, then $d = -A^{-2} - A^2$.

For a Type III Reidemeister move, we want the polynomial for $\langle \times \rangle$ to be the same as the polynomial for $\langle \times \rangle$.

$$\begin{aligned} \langle \text{crossing} \rangle &= A \langle \text{smooth} \rangle + A^{-1} \langle \text{smooth} \rangle \\ \text{applying a Type II move,} &= A(A \langle \text{smooth} \rangle + A^{-1} \langle \text{smooth} \rangle) = \langle \text{crossing} \rangle \end{aligned} \quad (2.2)$$

Thus, with these definitions of B and d , a Type III move has no effect on the polynomial.

Now, a Type I Reidemeister move is trickier. When we calculate the polynomial for $\langle \text{loop} \rangle$, we see that a Type I move changes the polynomial. To account for this, we introduce the notion of writhe. We give our knot an orientation by placing arrows on the knot. At each crossing, when we rotate the overstrand counterclockwise, if the arrows match up and point in the same direction, then that crossing gets assigned a $+1$ (Figure 2.1).



Figure 2.1: $+1$ crossing

If the arrows don't match up and instead point in opposite directions, then that crossing is assigned a -1 (Figure 2.2).



Figure 2.2: -1 crossing

The writhe of a knot is the sum of the numbers assigned to each crossing. Thus, depending on the orientation of the knot, when we perform a Type I Reidemeister move, we either increase or decrease the writhe by 1. Putting this together with our earlier calculations of how a Type I Reidemeister move changes the polynomial of a knot, we multiply the polynomial, denoted L calculated by the method described above by

$$(-A^3)^{-w(L)},$$

where $w(L)$ denotes the writhe of the knot to obtain a different polynomial, which we denote as K .

We need to make sure, however, that the writhe is unaffected by Type II and Type III Reidemeister moves. For a Type II Reidemeister move, note that the writhe is 0 since one crossing is $+1$ and the other is -1 . Thus, a Type II Reidemeister move is unaffected.



A Type III Reidemeister move is likewise, unaffected by the writhe, for moving a strand over or under a crossing, still gives us the same writhe since the strands are still oriented in the same way.



Thus, we have an invariant for knots and links. To better understand how to calculate the polynomial for a knot, as an example, we calculate the Kauffman polynomial for the trefoil.

$$\begin{aligned}
 \langle \text{Trefoil} \rangle &= A \langle \text{Crossing 1} \rangle + A^{-1} \langle \text{Crossing 2} \rangle \\
 &= A(A \langle \text{Crossing 1} \rangle + A^{-1} \langle \text{Crossing 1} \rangle) \\
 &\quad + A^{-1}(A \langle \text{Crossing 2} \rangle + A^{-1} \langle \text{Crossing 2} \rangle) \\
 &= A(A(A \langle \text{Crossing 1} \rangle + A^{-1} \langle \text{Crossing 1} \rangle) \\
 &\quad + A^{-1}(A \langle \text{Crossing 1} \rangle + A^{-1} \langle \text{Crossing 1} \rangle) \\
 &\quad + A^{-1}(A(A \langle \text{Crossing 2} \rangle + A^{-1} \langle \text{Crossing 2} \rangle) \\
 &\quad + A^{-1}(A \langle \text{Crossing 2} \rangle + A^{-1} \langle \text{Crossing 2} \rangle)) \\
 &= A^3 d^2 \langle \text{Circle} \rangle + Ad \langle \text{Circle} \rangle + Ad \langle \text{Circle} \rangle + A^{-1} \langle \text{Circle} \rangle \\
 &\quad + Ad \langle \text{Circle} \rangle + A^{-1} \langle \text{Circle} \rangle + A^{-1} \langle \text{Circle} \rangle + A^{-3} d \langle \text{Circle} \rangle \\
 &= A^3(A^4 + 2 + A^{-4}) + 3A(-A^2 - A^{-2}) \\
 &\quad + 3A^{-1} + A^{-3}(-A^2 - A^{-2}) \\
 &= A^7 - A^3 - A^{-5} \tag{2.3}
 \end{aligned}$$

Note that if we place an orientation on the trefoil, we have a writhe of -3 (Figure 2.3).

Thus, our final polynomial for the trefoil with this particular orientation is $\langle K \rangle = (A^7 - A^3 - A^{-5})(-A^3)^3 = -A^{16} + A^{12} + A^4$

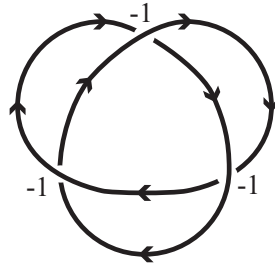


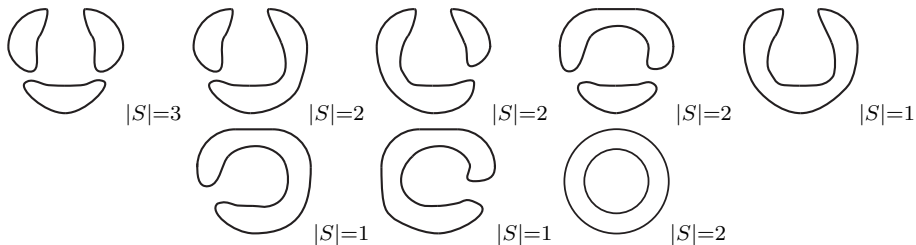
Figure 2.3: Oriented trefoil

However, it's not clear that this is well-defined. After all, how do we know which crossing to begin with when we make our first A and B splits and does it make a difference in the final polynomial of the knot? What if we had chosen a different crossing for our first split in the trefoil?

The bracket polynomial gives us the same polynomial calculated using our earlier method, but in such a way that it's clear that it's well-defined[1]. For every knot, we have a choice of how we wish to split each crossing, either an A split or a B split, so that the resulting projection has no crossings. Each choice is called a state. The bracket polynomial is calculated based on all the possible states of a knot. For each state, let $a(s)$ be the number of A splits in our choice and $b(s)$ be the number of B splits in our state. Define $|S|$ to be the number of resulting unknots. The formula for the bracket polynomial is given by

$$\sum_S A^{a(s)} A^{-b(s)} (-A^2 - A^{-2})^{|S|-1}.$$

Since we're considering all possible ways to dissolve the crossings of a knot, it's clear that the bracket polynomial is well defined, that no matter which crossing we choose first, we eventually consider all of them. As an example, we calculate the trefoil knot using the formula for the bracket polynomial.



$$\begin{aligned}
\langle K \rangle &= A^3 A^0 (-A^2 - A^{-2})^{3-1} + A^2 A^{-1} (-A^2 - A^{-2})^{2-1} \\
&\quad + A^2 A^{-1} (-A^2 - A^{-2})^{2-1} + A^2 A^{-1} (-A^2 - A^{-2})^{2-1} \\
&\quad + A^1 A^{-2} (-A^2 - A^{-2})^{1-1} + A^1 A^{-2} (-A^2 - A^{-2})^{1-1} \\
&\quad + A^1 A^{-2} (-A^2 - A^{-2})^{1-1} + A^0 A^{-3} (-A^2 - A^{-2})^{2-1} \\
&= A^3 (-A^2 - A^{-2})^2 + 3A (-A^2 - A^{-2}) + 3A^{-1} + A^{-3} (-A^2 - A^{-2}) \\
&= A^7 - A^3 - A^{-5} \tag{2.4}
\end{aligned}$$

Note that the bracket polynomial for the trefoil is the same as the Jones polynomial for the trefoil. In fact, for any knot we will get the same polynomial despite how we calculate it. When we calculate the Jones polynomial, we begin with a crossing and we consider both the resulting A splitted projection and the resulting B splitted projection. We then continue to dissolve the crossings in each of the projections. Note that all the possible states of the trefoil appear in the last line of the calculations of the Kauffman polynomial for the trefoil, with the appropriate powers of A , B , and d .

When we make the substitution $A = t^{-\frac{1}{4}}[1]$ for the Kauffman polynomial, we obtain the Jones polynomial. Thus, the Jones polynomial and the Kauffman polynomial of a knot are the same and the Kauffman polynomial is just as powerful a knot invariant as the Jones polynomial, but with a more intuitive combinatorial method of calculation. Throughout this thesis, we will calculate polynomials primarily using the Kauffman polynomial, and then making the necessary substitutions at the end.

We've already seen the Kauffman polynomial for a trefoil, which is also a $(2, 3)$ torus knot, but what of the other torus knots? Vaughan Jones[1] has proven the following formula for an (m, n) torus knot.

$$\langle K_{(m,n)} \rangle = \frac{t^{\frac{(m-1)(n-1)}{2}} (1 - t^{m-1} - t^{n-1} + t^{(m+n)})}{(1 - t^2)} \tag{2.5}$$

The proof of this, however, relies on difficult algebraic methods. We wish to find a simple proof of this fact by means of induction.

Recall that we can consider each torus knot as the closure of a braid, represented by a braid word. When we make A and B splits in these braids, we no longer necessarily have a braid for we may have loops that double back on itself. We call these tangles. Nonetheless, just as we did with a knot, we can consider the state of a braid after we make particular choices of A and B splits for each crossing. These elements form an algebra similar to the braids, with multiplication as the associative binary operation. The

identity element is the same as that of the braid group. This is known as the Temperley-Lieb algebra. The Temperley-Lieb Algebra for n strands is generated by n elements[4]. The identity, which is the braid of n strands with no crossings, and a set of U_1, U_2, \dots, U_{n-1} 's such that each U_i represents the B split of a σ_i crossing (Figure 2.4).

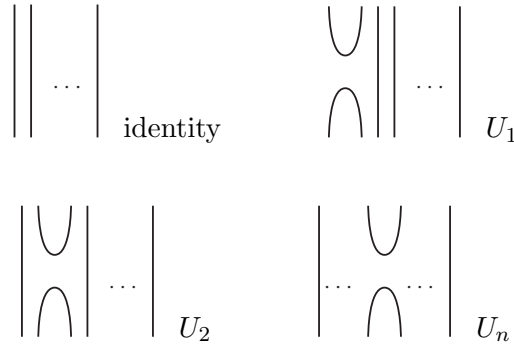
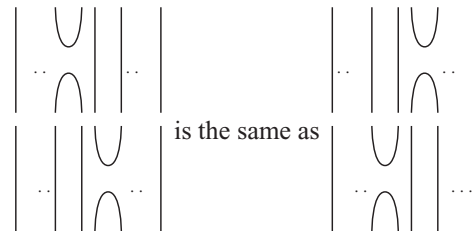


Figure 2.4: Temperley-Lieb Algebra generators

The Temperley-Lieb Algebra for n strands behaves similarly to the Artin braid group for n strands. Multiplication is defined in the same manner, and is commutative if and only if for $U_i, U_j, |j - i| \leq 1$. However, the Temperley-Lieb Algebra of n strands also has some other nice properties. Each element is its own identity multiplied by a constant. Furthermore, $U_i U_{i\pm 1} U_i = U_i$. These properties are summed below:

Property 1 $U_i U_j = U_j U_i$, if $|j - i| \leq 1$



Property 2 $U_i U_{i\pm 1} U_i = U_i$

$$\left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \\ \vdots \end{array} \right| = \left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \\ \vdots \end{array} \right|$$

Property 3 $U_i U_i = k U_i$, where k is a constant.

$$\left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \\ \vdots \end{array} \right| = \left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \\ \vdots \end{array} \right| = d \left| \begin{array}{c} \vdots \\ \text{---} \\ \vdots \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \\ \vdots \end{array} \right|$$

To better understand the Temperley-Lieb Algebra, we take a look at the Temperley-Lieb Algebra on 3 strands. The generators for this group are U_1 , U_2 , and the identity, and the elements generated by these three elements include $U_1 U_2$ and $U_2 U_1$ (Figure 2.5).

Note that the 3 properties of the Temperley-Lieb Algebra apply, but since we only have two elements which are not the identity, the first property is irrelevant. Note that each element is its own identity. For the identity, $U_1 U_2$, and $U_2 U_1$, the constant is 1. For U_1 and U_2 , however, the constant is $d = -A^{-2} - A^2$, since we can pull out an unknot when we multiply U_1 and U_2 with itself.

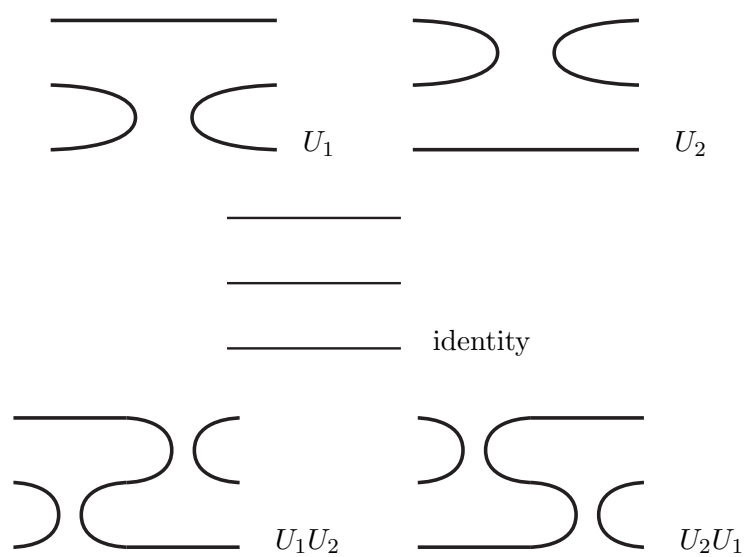


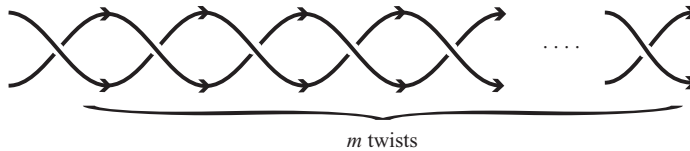
Figure 2.5: Elements of the Temperley-Lieb Algebra on 3 strands

Chapter 3

$$n = 2$$

For the $n = 2$ case, we choose to look at the $(m, 2)$ torus knot as a braid. We consider all the possible states of the braid and then reconnect the ends to obtain our torus knot, adjusting for constants as necessary when we reconnect. Using the definition for calculating the bracket polynomial, if we find the individual polynomial for each state of the braid, and then reconnect the ends, we will have the individual polynomials for each state of the knot. We can add up these state polynomials to obtain the Jones polynomial for the knot.

Our $(m, 2)$ torus knot, in unconnected braid form, would have two strands with m twists and look something like this:



Since the Temperley-Lieb Algebra on 2 strands contains only two elements, the identity and U_1 , we only have two states to consider.



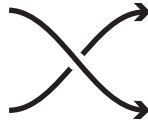
Thus, we need only to calculate the polynomial for each state. In other words, for an $(m, 2)$ torus knot, we need to determine the number of A splits or B splits necessary to obtain each element of the Temperley-Lieb Algebra on 2 strands. We temporarily label the polynomial for the identity state

of an $(m, 2)$ torus knot as P_m , and the polynomial for the U_1 state of the $(m, 2)$ as Q_m .

By induction on the number of twists, we will show that the P_m and Q_m polynomial for an unattached $(m, 2)$ torus knot are

$$\begin{aligned} P_m &= A^m \\ Q_m &= \frac{A^{m-2}(1 - (-1)^m A^{-4m})}{1 + A^{-4}}. \end{aligned}$$

We start with the base case and consider the unattached two stranded torus knot with only one twist.

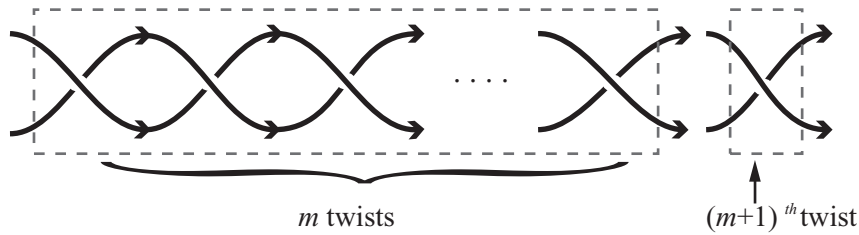


It is easy to see that since we only have one crossing, we only have two possible states: the state when we make an A split and the state when we make a B split. When we make an A split, we get the identity element, and making a B split gives us the U_1 element. Thus, the polynomials are for the two stranded unattached torus knot with one twist are

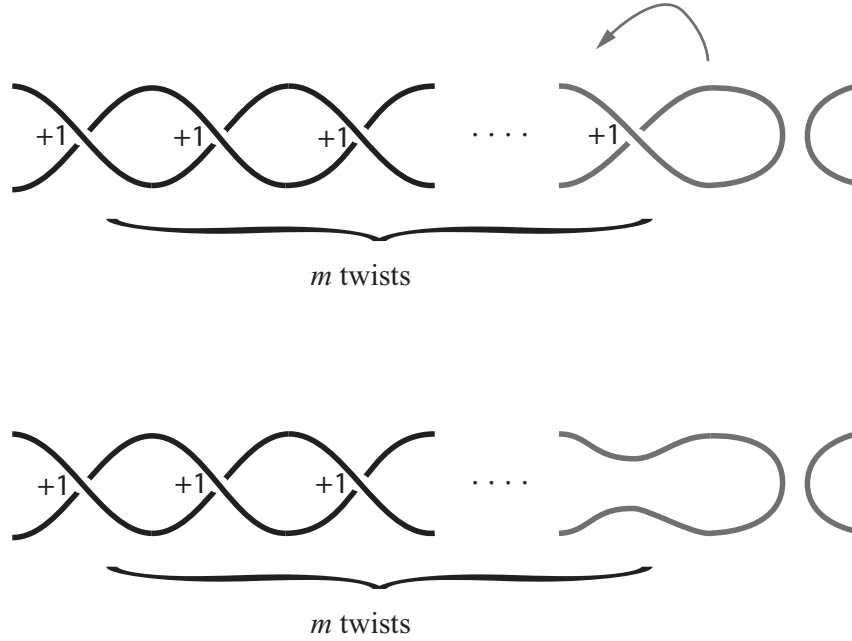
$$P_1 = A \text{ and } Q_1 = A^{-1}.$$

Thus, substituting $m = 1$ for Q_m and simplifying, we have proven our polynomial for when $m = 1$.

For the inductive step, we assume that it's true for m twists, and we shall prove it for $m + 1$ twists. Thus, we have something that looks like the following diagram.



The $(m + 1)^{th}$ twist can be split in two ways: A split and B split. Note that when we make a B split, then pinning the loose ends down, we can twist the braid m times to remove each crossing and obtaining the non-identity element for the Temperley-Lieb Algebra on two strands, U_i .



Since we're only interested in the torus knot, $m + 1$ must be relatively prime to 2; that is, $m + 1$ must be odd. Considering an odd number of twists, then no matter how we choose to place the orientation on the closed braid, or the $(m, 2)$ torus knot, each time we untwist, after making a B split, we're removing a $+1$ crossing. Since for every torus knot, each crossing would be positively oriented, we assume that even when m is not relatively prime to 2 and it's an unattached link, each crossing is still positively oriented. Thus, we must multiply our polynomial by $(-A^3)^{-m}$, since we're performing m Type I moves on m positive crossings. Thus, we have

$$(A^{-1})(-A^3)^{-m}.$$

Simplifying,

$$(A^{-1})(-A)^{-3m} = (-1)^m A^{-3m-1}$$

When we make an A split, we have the unattached torus knot of two strands with m twists. Thus, we have

$$A[P_m + Q_m]$$

Only P_m gives us the identity element of the Temperley-Lieb Algebra on 2 strands. Thus, by the inductive hypothesis,

$$P_{m+1} = A(P_m) = A(A^m) = A^{m+1}$$

Now, $A(Q_m)$ gives us the other element of the Temperley-Lieb Algebra on 2 strands, U_1 . Since making a B split gives us this element as well when we decompose all crossings, we add the two resulting polynomials together. Thus,

$$\begin{aligned}
Q_{m+1} &= A(Q_m) + A^{-3m-1} \\
&= A \left[\frac{A^{m-2}(1 - (-1)^m(A^{-4m}))}{1 + A^{-4}} \right] + (-1)^m(A^{-3m-1}) \\
&= \frac{1}{1 + A^{-4}} [A^{m-1}(1 - (-1)^m A^{-4m}) + (-1)^m(A^{-3m-1})(1 + A^{-4})] \\
&= \frac{1}{1 + A^{-4}} [A^{m-1} + (-1)^m A^{-3m-5}] \\
&= \frac{1}{1 + A^{-4}} [A^{m-1} - (-1)^{m+1} A^{-3m-5}] \\
&= \frac{A^{m-1}}{1 + A^{-4}} [1 - (-1)^{m+1} A^{-4m-4}] \\
&= \frac{A^{(m+1)-2}(1 - (-1)^{m+1} A^{-4(m+1)})}{1 + A^{-4}} \tag{3.1}
\end{aligned}$$

We have proven our claim for $m + 1$.

Thus, the polynomials for each decomposed state of the unattached torus knot of two strands with m twists are

$$P_m = A^m \text{ and } Q_m = \frac{A^{m-2}(1 - (-1)^m(A^{-4m}))}{1 + A^{-4}}$$

for the identity element and the U_1 element respectively of the Temperley-Lieb Algebra on 2 strands.

To calculate the polynomial for the torus knot, which is what we are interested in, we must reattach the ends of each element of the Temperley-Lieb Algebra. Note that for the element U_1 , when we reattach the ends to consider this decomposed state of the $(m, 2)$ torus knot, we only obtain one unknot. Since we know the polynomial for the unknot is just 1, we have no extra constants. Thus, the polynomial for this decomposed state of the $(m, 2)$ torus knot is exactly

$$Q_m = \frac{A^{m-2}(1 - (-1)^m(A^{-4m}))}{1 + A^{-4}}.$$

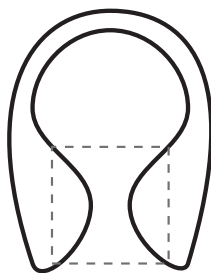
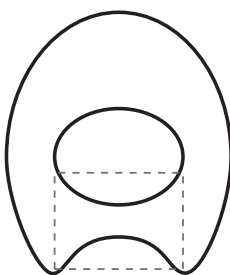
Figure 3.1: U_1 element reattached

Figure 3.2: Identity element reattached

When we reattach the ends of the identity element, we obtain two unknots, and thus, an extra constant, d . Hence, the polynomial for this decomposed state of the $(m, 2)$ torus knots is

$$d(P_m) = (-A^{-2} - A^2)(A^m) = -A^{m-2} - A^{m+2}$$

Recall that the Jones polynomial for a knot is the sum of the polynomials of all its decomposed states. Thus, defining $K_{(m,n)}$ to be the polynomial for an (m, n) torus knot,

$$\langle K_{(m,2)} \rangle = d(P_m) + Q_m = -A^{m-2} - A^{m+2} + \frac{A^{m-2}(1 - (-1)^m(A^{-4m}))}{1 + A^{-4}}$$

Notice that with an $(m, 2)$ torus knot, we have a writhe of $(-A^3)^{-m}$, since we have m crossings. As we noted before, in order to have a knot instead of a link, m must be odd, so $(-A^3)^{-m} = -A^{-3m}$, and $(-1)^m = -1$

Thus, the Jones polynomial for the $(m, 2)$ torus knot is

$$\langle K_{(m,2)} \rangle = \left[(-A^{m-2} - A^{m+2}) + \frac{A^{m-2}(1 + (A^{-4m}))}{1 + A^{-4}} \right] (-A^{-3m})$$

Substituting A for $t^{-\frac{1}{4}}$ and factoring, we have

$$\begin{aligned}
\langle K_{(m,2)} \rangle &= \frac{-t^{\frac{3m}{4}}}{1+t} \left[(-t^{-\frac{m+2}{4}} - t^{-\frac{m-2}{4}})(1+t) + t^{-\frac{m+2}{4}}(1+t^m) \right] \\
&= \frac{1}{1+t} \left[t^{\frac{m+1}{2}} + t^{\frac{m-1}{2}} + t^{\frac{m+3}{2}} + t^{\frac{m+1}{2}} - t^{\frac{m+1}{2}} - t^{\frac{3m+1}{2}} \right] \\
&= \frac{1}{1+t} \left[t^{\frac{m+1}{2}} + t^{\frac{m-1}{2}} + t^{\frac{m+3}{2}} - t^{\frac{3m+1}{2}} \right] \\
&= \frac{t^{\frac{m-1}{2}}(-t-1-t^2+t^{m+1})(1-t)}{1-t^2} \\
&= \frac{t^{\frac{m-1}{2}}(1-t^{m+1}-t^3+t^{m+2})}{1-t^2}
\end{aligned} \tag{3.2}$$

We see that this is exactly Vaughan Jones' formula for an $(m, 2)$ torus knot.

Chapter 4

$$n = 3$$

For the $(m, 3)$ torus knot, we employ the same techniques and methods that we used for the $n = 2$ case. Recall that the generators for the Temperley-Lieb Algebra on 3 strands are id (identity), U_1 , and U_2 . These elements generate the other two elements of the Temperley-Lieb Algebra on 3 strands: U_1U_2 and U_2U_1 . Thus, we have five elements in the Temperley-Lieb Algebra on 3 strands. As with the $n = 2$ case, we consider the unattached 3 stranded braid of the $(m, 3)$ torus knot.

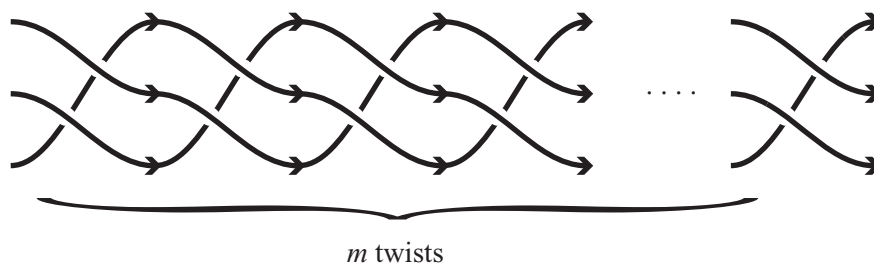
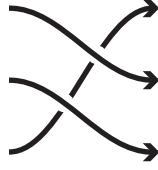


Figure 4.1: $(m, 3)$ torus knot braid representation

Note that with each twist, we have a writhe of -2 . Thus, the writhe of an $(m, 3)$ torus knot is $(-A^{-3})^{-2m} = A^{-6m}$.

We will temporarily assign P_m as the polynomial that decomposes the unattached $(m, 3)$ braid into the identity element, Q_m as the polynomial for the U_1 element, R_m for U_2 , S_m for U_1U_2 , and T_m for U_2U_1 . We first consider the braid with only one twist, that is $m = 1$.



Since each twist has two crossings, and two possible splits for each crossing, we have four possibilities. We could make an A split in both crossings, which would give us the identity element of the Temperley-Lieb Algebra for three strands. Thus,

$$P_1 = A^2. \quad (4.1)$$

Another possibility is to make a B split in the top crossing and an A split in the bottom crossing, which would give us the U_1 element, and a polynomial of 1.

Hence,

$$Q_1 = 1. \quad (4.2)$$

Another choice is to make an A split in the the crossing and a B split in the bottom crossing, which gives us the U_2 and a polynomial of 1 as well. So,

$$R_1 = 1. \quad (4.3)$$

Finally, if we make two B splits, we obtain U_1U_2 . Thus,

$$S_1 = A^{-2}. \quad (4.4)$$

Notice, however, that no matter what combination of A splits or B splits are we able to get T_1 . So, we let

$$T_1 = 0. \quad (4.5)$$

Using induction on the number of twists, we will consider the $m \equiv 1 \pmod{3}$ case and the $m \equiv 2 \pmod{3}$ separately.

For the $m \equiv 1 \pmod{3}$ case, I claim that

$$\begin{aligned} P_m &= A^{2m}, \\ Q_m &= \frac{1}{1 - A^{-12}}(A^{2m-2} - A^{-2m-10} - A^{2m-10} + A^{-2m-6}), \\ R_m &= \frac{1}{1 - A^{-12}}(A^{2m-2} - A^{-2m-10} - A^{2m-10} + A^{-2m-6}), \\ S_m &= \frac{1}{1 - A^{-12}}(A^{2m-4} - A^{-2m-12} - A^{2m-8} + A^{-2m-4}), \\ T_m &= \frac{1}{1 - A^{-12}}(A^{2m-4} - A^{-2m} - A^{2m-8} + A^{-2m-4}). \end{aligned}$$

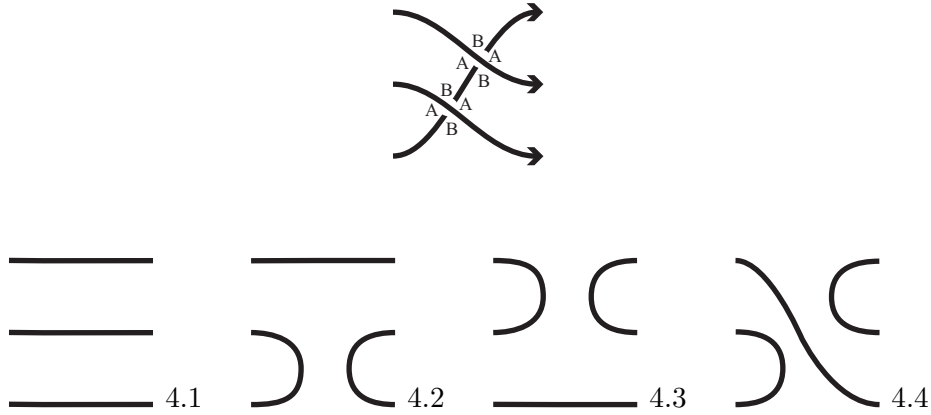


Figure 4.2: Polynomials for unattached (1, 3) torus knot

It is easy to check that this holds for the $m = 1$ case since

$$\begin{aligned}
 P_1 &= A^{2m} = A^{2(1)} = A^2, \\
 Q_1 &= \frac{1}{1 - A^{-12}}(1 - A^{-12}) = 1, \\
 R_1 &= \frac{1}{1 - A^{-12}}(1 - A^{-12}) = 1, \\
 S_1 &= \frac{1}{1 - A^{-12}}(A^{-2} - A^{-14}) = A^{-2}, \text{ and} \\
 T_1 &= \frac{0}{1 - A^{-12}}.
 \end{aligned}$$

Note that the braid representation of the $(m, 3)$ torus knot is just the braid representation of the $(m-1, 3)$ torus knot adjoined with the $(1, 3)$ torus knot. Since all our braid are eventually closed, it does not matter if we are adjoining the extra twist on the left or on the right. Nonetheless, when we start considering the decomposed states of that particular twist, which are the elements of the Temperley-Lieb Algebra, we need to be consistent as to whether we are adjoining on the left or on the right since the elements of the Temperley-Lieb Algebra is not commutative. For our purposes, we consider adjoining the twist on the right.

Since it does not matter for the Jones polynomial the order in which we choose to decompose the crossings, we can think of adjoining each additional twist as right multiplication of the $P_{m-1}, Q_{m-1}, R_{m-1}, S_{m-1}, T_{m-1}$ by the P_1, Q_1, R_1, S_1, T_1 polynomials. In other words, the polynomial P_m can be

obtained by the following formula

$$P_m = P_{m-1}P_1 \tag{4.6}$$

since the only way we get an identity element for the braid representation of the $(m, 3)$ torus knot is if adjoining the braid representation of the $(m - 1, 3)$ torus knot and the braid representation of the $(1, 3)$ torus knot, we obtain the identity element. However, this only occurs if $(m - 1, 3)$ decomposed into the identity element, which gives us the P_{m-1} polynomial, is adjoined on the right with the identity element of the $(1, 3)$ twist, which gives us the P_1 polynomial (See Figure 4.3).

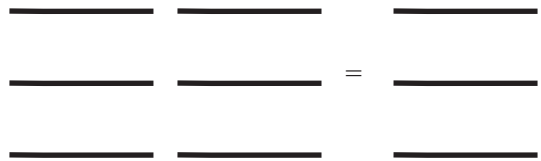


Figure 4.3: Identity adjoined identity

Since $P_1 = A^2$, we can conclude inductively that $P_m = A^{2m}$.

We now consider the other elements of the Temperley-Lieb Algebra on 3 strands.

The polynomial for the U_1 element is represented by Q_m . When multiplying two elements together, we get U_1 when we multiply U_1 with the identity element. Even though the identity commutes in the Temperley-Lieb Algebra, because the $U_1(id)$ represents Q_{m-1} polynomial multiplied by P_1 (since we are adjoining our twists on the right) and $(id)U_1$ represents the $P_{m-1}Q_1$ polynomial. We also know that U_1U_1 gives us the U_1 by the properties of the Temperley-Lieb Algebra. However, note that when we multiply the U_1 element by itself, we get an extra constant, $d = -A^2 - A^{-2}$.

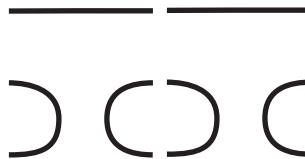


Figure 4.4: U_1U_1

Recalling another property of the Temperley-Lieb Algebra, we note that multiplying U_1 on the right by U_2U_1 also gives us U_1 with no extra constants, which is represented by the multiplication of the polynomials, Q_{m-1}

and T_1 . Also, $(U_1U_2)U_1 = U_1$ as well with no extra constants; this is represented by $S_{m-1}Q_1$. Furthermore, $U_1U_2U_2U_1$ also gives us U_1 with an extra constant $d = -A^2 - A^{-2}$, represented by the polynomial $S_{m-1}T_1$. Figure 4.5 demonstrates these relations.

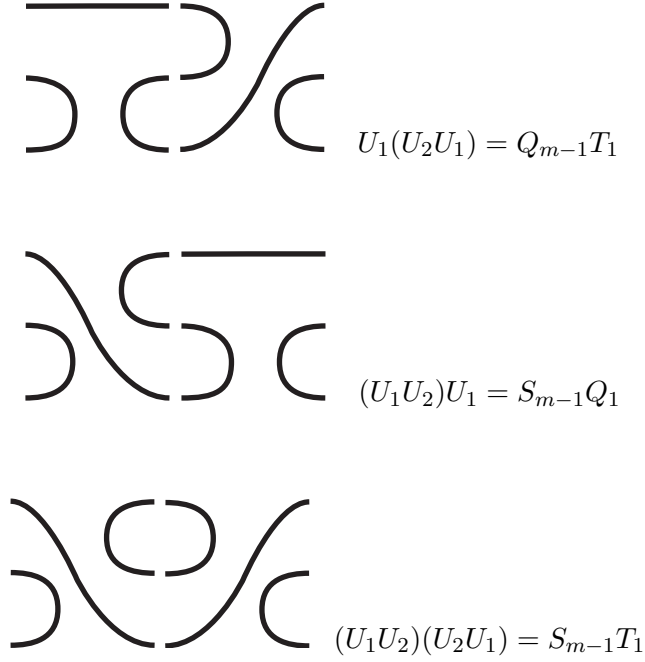


Figure 4.5: Relations of elements of Temperley-Lieb Algebra on 3 strands

Thus,

$$\begin{aligned}
 Q_m &= P_{m-1}Q_1 + Q_{m-1}P_1 + Q_{m-1}Q_1(d) \\
 &\quad + Q_{m-1}T_1 + S_{m-1}T_1(d) + S_{m-1}Q_1.
 \end{aligned} \tag{4.7}$$

Using the same methods, we can find a similar formula for R_m, S_m, T_m .

$$\begin{aligned}
 R_m &= P_{m-1}R_1 + R_{m-1}P_1 + R_{m-1}R_1(d) \\
 &\quad + R_{m-1}S_1 + T_{m-1}S_1(d) + T_{m-1}R_1,
 \end{aligned} \tag{4.8}$$

$$\begin{aligned}
 S_m &= P_{m-1}S_1 + S_{m-1}P_1 + Q_{m-1}S_1(d) \\
 &\quad + Q_{m-1}R_1 + S_{m-1}R_1(d) + S_{m-1}S_1,
 \end{aligned} \tag{4.9}$$

$$\begin{aligned}
 T_m &= P_{m-1}T_1 + T_{m-1}P_1 + R_{m-1}T_1(d) \\
 &\quad + R_{m-1}Q_1 + T_{m-1}Q_1(d) + T_{m-1}T_1.
 \end{aligned} \tag{4.10}$$

Before proving the $m \equiv 1 \pmod{3}$ case, we prove the $m \equiv 2 \pmod{3}$ by assuming that the $m - 1 \equiv 1 \pmod{3}$ is true. That is,

$$\begin{aligned} P_{m-1} &= A^{2m-2}, \\ Q_{m-1} &= \frac{1}{1-A^{-12}}(A^{2m-4} - A^{-2m-8} - A^{2m-12} + A^{-2m-4}), \\ R_{m-1} &= \frac{1}{1-A^{-12}}(A^{2m-4} - A^{-2m-8} - A^{2m-12} + A^{-2m-4}), \\ S_{m-1} &= \frac{1}{1-A^{-12}}(A^{2m-6} - A^{-2m-10} - A^{2m-10} + A^{-2m-2}), \\ T_{m-1} &= \frac{1}{1-A^{-12}}(A^{2m-6} - A^{-2m+2} - A^{2m-10} + A^{-2m-2}). \end{aligned}$$

Using Equation 4.7, with some substitutions and algebraic manipulations, we have:

$$Q_m = \frac{1}{1-A^{-12}}(A^{2m-2} - A^{-2m-6} - A^{2m-10} + A^{-2m-2}).$$

Thus, when $m \equiv 2 \pmod{3}$, Q_m is the above equation. With the same method, we derive R_m , S_m and T_m for when $m \equiv 2 \pmod{3}$.

$$\begin{aligned} R_m &= \frac{1}{1-A^{-12}}(A^{2m-2} - A^{-2m-6} - A^{2m-10} + A^{-2m-2}), \\ S_m &= \frac{1}{1-A^{-12}}(A^{2m-4} - A^{-2m-8} - A^{2m-8} + A^{-2m-12}), \\ T_m &= \frac{1}{1-A^{-12}}(A^{2m-4} - A^{-2m-8} - A^{2m-8} + A^{-2m}). \end{aligned}$$

Now, knowing the formulas for when $m \equiv 2 \pmod{3}$, we will find the formulas in the same way for when $m + 1 \equiv 0 \pmod{3}$. Though when m is a multiple of 3, we obtain a link and not a torus knot, the formulas are necessary to derive the formulas for when $m + 2 \equiv 1 \pmod{3}$ as we inductively form each formula. So, using the same method and formulas as before, substituting m for $m + 1$, we have the following for when $m + 1 \equiv 0 \pmod{3}$:

$$\begin{aligned} P_{m+1} &= A^{2m+2}, \\ Q_{m+1} &= \frac{1}{1-A^{-12}}(A^{2m} - A^{-2m-4} - A^{2m-8} + A^{-2m-12}), \\ R_{m+1} &= \frac{1}{1-A^{-12}}(A^{2m} - A^{-2m-4} - A^{2m-8} + A^{-2m-12}), \end{aligned}$$

$$\begin{aligned}
S_{m+1} &= \frac{1}{1-A^{-12}}(A^{2m-2} - A^{-2m-8} - A^{2m-6} + A^{-2m-10}), \\
T_{m+1} &= \frac{1}{1-A^{-12}}(A^{2m-2} - A^{-2m-6} - A^{2m-6} + A^{-2m-10}).
\end{aligned}$$

Now that we have the formulas for when $m+1 \equiv 0 \pmod{3}$, we use the same method to find $m+2 \equiv 1 \pmod{3}$:

$$\begin{aligned}
P_{m+2} &= A^{2m+4}, \\
Q_{m+2} &= \frac{1}{1-A^{-12}}(A^{2m+2} - A^{-2m-14} - A^{2m-6} + A^{-2m-10}), \\
R_{m+2} &= \frac{1}{1-A^{-12}}(A^{2m+2} - A^{-2m-14} - A^{2m-6} + A^{-2m-10}), \\
S_{m+2} &= \frac{1}{1-A^{-12}}(A^{2m} - A^{-2m-16} - A^{2m-4} + A^{-2m-8}), \\
T_{m+2} &= \frac{1}{1-A^{-12}}(A^{2m} - A^{-2m-4} - A^{2m-4} + A^{-2m-8}).
\end{aligned}$$

With some algebra, we can easily see that this is the formula for $m \equiv 1 \pmod{3}$, where we substitute m for $m+2$. By this method of induction, we have proven our claim. We observe within each case, Q_m always equals R_m .

To check our formulas with the known formula for the closed form of the Jones polynomial, we consider the closing of each element of the Temperley-Lieb Algebra. Note that when we close the identity, we have three unknots. Pulling two of them out, we have d^2 . For the U_1 element, the closing has two unknots, which gives us an extra d , and likewise for the U_2 element. The closing of the U_1U_2 and U_2U_1 elements have no extra unknots. Thus,

$$\langle K_{(m,3)} \rangle = d^2(P_m) + Q_m d + R_m d + S_m + T_m. \quad (4.11)$$

Since we have formulas for when $m \equiv 1 \pmod{3}$ and when $m \equiv 2 \pmod{3}$, we consider them separately, disregarding the links we obtain when m is a multiple of 3. When $m \equiv 1 \pmod{3}$, we have

$$\begin{aligned}
\langle K_{(m,3)} \rangle &= \frac{1}{1-A^{-12}} \left[(A^4 + 2 + A^{-4})(A^{2m})(1 - A^{-12}) \right. \\
&\quad + (-2A^2 - 2A^{-2})(A^{2m-2} - A^{-2m-10} - A^{2m-10} + A^{-2m-6}) \\
&\quad + (A^{2m-4} - A^{-2m-12} - A^{2m-8} + A^{-2m-4}) \\
&\quad \left. + (A^{2m-4} - A^{-2m} - A^{2m-8} + A^{-2m-4}) \right]. \quad (4.12)
\end{aligned}$$

Note that for each twist, the writhe is $+2$, e.g. we multiply 4.12 by $(-A^3)^{-2m} = A^{-6m}$ since the writhe will always be even. After simplifying, we have

$$\begin{aligned} \langle K_{(m,3)} \rangle = & \frac{1}{1 - A^{-12}} \left[A^{-4m+4} - A^{-4m-16} + A^{-8m-12} \right. \\ & \left. + A^{-4m-4} - A^{-8m} - A^{-4m-8} \right]. \end{aligned} \quad (4.13)$$

Substituting $A = t^{\frac{-1}{4}}$, and with some algebraic manipulations, we have exactly Vaughan Jones' formula 2.5 for the $(m, 3)$ torus knot:

$$\langle K_{(m,3)} \rangle = \frac{t^{m-1}(1 - t^{m+1} - t^4 + t^{m+3})}{1 - t^2}. \quad (4.14)$$

For the $m \equiv 2 \pmod{3}$, we have

$$\begin{aligned} \langle K_{(m,3)} \rangle = & \frac{1}{1 - A^{-12}} \left[(A^4 + 2 + A^{-4})(A^{2m})(1 - A^{-12}) \right. \\ & + (-2A^2 - 2A^{-2})(A^{2m-2} - A^{-2m-6} - A^{2m-10} + A^{-2m-2}) \\ & + (A^{2m-4} - A^{-2m-8} - A^{2m-8} + A^{-2m-12}) \\ & \left. + (A^{2m-4} - A^{-2m-8} - A^{2m-8} + A^{-2m}) \right] \end{aligned} \quad (4.15)$$

It is easy to check that after simplifying and multiplying 4.15 by the writhe, which is $(-A^3)^{-w(L)} = A^{-6m}$, we have equation 4.13, which we have already shown is Vaughan Jones' formula when we make the appropriate substitutions.

Chapter 5

Weaving Knots

Now that we have explored the $(m, 2)$ and $(m, 3)$ torus knots, it is easy to see how we can apply this method to any (m, n) torus knot by finding the polynomials for the decomposed forms of the $(1, n)$ torus knot and the relations of the Temperley-Lieb Algebra on n strands. This combinatorial method for finding the polynomial of an (m, n) torus knot can also be applied to any family of knots generated by a repeated pattern of twists. For example, we look specifically at another family of knots known as weaving knots.

An (m, n) weaving knot represented in braid form looks similar to an (m, n) torus knot except with alternating crossings (See Figure 5.1).

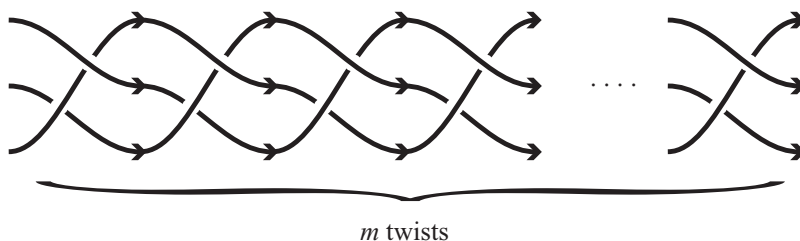


Figure 5.1: An $(m, 3)$ weaving knot

Note that the $(m, 2)$ weaving knot is the $(m, 2)$ torus knot since the crossings of the $(m, 2)$ torus knot are already alternating in its representative braid form. Furthermore, since the knot is alternating, the manner in which we choose our crossings is not particularly important as long as we are consistent. If we choose to pick the first crossing to be a $+1$ crossing instead of a -1 crossing or vice versa, we merely need to change all our exponents by a negative sign since changing an overcrossing to an undercrossing changes

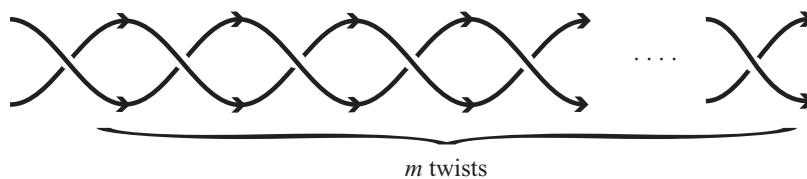
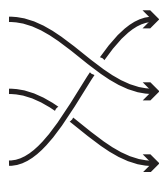


Figure 5.2: A $(m, 2)$ torus knot is also a $(m, 2)$ weaving knot.

the A splits to B splits and vice versa. For consistency, we choose our top strand or the n^{th} strand to lie over the $(n - 1)^{th}$ strand. That is, our braid word should end with σ_{n-1}^{-1} .

Since we are well aware of the polynomial for the $(m, 2)$ weaving knot, we explore the polynomials for an $(m, 3)$ weaving knot. As before, we use the same method of examining the polynomials to decompose one twist in our weaving knot into the elements of the Temperley-Lieb Algebra on 3 strands and then using the relations of the elements to iterate the polynomials for the $(m, 3)$ weaving knot.



Since we already know the relations of the Temperley-Lieb Algebra on 3 strands, we can write a program to generate the $(m, 3)$ weaving knot or $(m, 3)$ torus knot by merely changing the initial conditions. That is, decomposing one twist of the $(m, 3)$ weaving knot will give us different polynomials for each element of the Temperley-Lieb Algebra than decomposing one twist of the $(m, 3)$ torus knot, but the relations of the Temperley-Lieb Algebra on 3 strands will still hold. In other words, $U_1 U_1$ still equals U_1 multiplied by a constant, even if Q_m , which represents the polynomial for U_1 , is different. Thus, if we find the polynomials for each unattached state of the $(1, 3)$ weaving knot, we can iterate the polynomial for the $(m, 3)$ weaving knot.

Recall that the relations of the Temperley-Lieb Algebra on 3 strands for m twists are

$$\begin{aligned}
 P_m &= P_{m-1}P_1, \\
 Q_m &= P_{m-1}Q_1 + Q_{m-1}P_1 + Q_{m-1}Q_1(d) \\
 &\quad + Q_{m-1}T_1 + S_{m-1}T_1(d) + S_{m-1}Q_1,
 \end{aligned}$$

$$\begin{aligned}
 R_m &= P_{m-1}R_1 + R_{m-1}P_1 + R_{m-1}R_1(d) \\
 &\quad + R_{m-1}S_1 + T_{m-1}S_1(d) + T_{m-1}R_1, \\
 S_m &= P_{m-1}S_1 + S_{m-1}P_1 + Q_{m-1}S_1(d) \\
 &\quad + Q_{m-1}R_1 + S_{m-1}R_1(d) + S_{m-1}S_1, \\
 T_m &= P_{m-1}T_1 + T_{m-1}P_1 + R_{m-1}T_1(d) \\
 &\quad + R_{m-1}Q_1 + T_{m-1}Q_1(d) + T_{m-1}T_1,
 \end{aligned}$$

where $d = -A^2 - A^{-2}$, and P_m represents the polynomial for the identity after decomposing m twists on 3 strands, Q_m represents the polynomial for the U_1 element, R_m is the polynomial for U_2 , S_m represents the polynomial for U_1U_2 , and T_m represents U_2U_1 .

Decomposing one twist of the $(m, 3)$ weaving knot, we find that

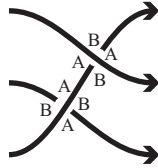
$$P_1 = 1, \tag{5.1}$$

$$Q_1 = A^2, \tag{5.2}$$

$$R_1 = A^{-2}, \tag{5.3}$$

$$S_1 = 1, \tag{5.4}$$

$$T_1 = 0. \tag{5.5}$$



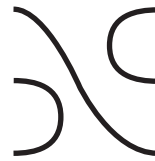
5.1



5.2



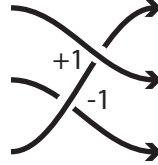
5.3



5.4

Note that the closing formulas for each element of the Temperley-Lieb Algebra still remain the same. Specifically, when we join the ends of the identity element, we get 2 extra unknots, and so just as before, the P_m polynomial is multiplied by d^2 with the ends are reattached. Recall that for the $(m, 3)$ torus knot, with each twist, we increased the writhe of the knot by $+2$. With the $(m, 3)$ weaving knot, however, with each twist, the writhe is exactly 0 since in each twist of the $(m, 3)$ weaving knot we have a

positive crossing and a negative crossing. Thus, the polynomial for the $(m, 3)$ weaving knot is just the addition of all the polynomials of the reattached forms of each element of the Temperley-Lieb Algebra on 3 strands.



The following are the polynomials for the $(m, 3)$ weaving knot where $m = 2, 4, 5, 7$.

$$\langle W_{(2,3)} \rangle = A^8 - A^4 + 1 - A^{-4} + A^{-8} \quad (5.6)$$

$$\langle W_{(4,3)} \rangle = A^{16} - 4A^{12} + 6A^8 - 7A^4 + 9 - 7A^{-4} + 6A^{-8} - 4A^{-12} + A^{-16} \quad (5.7)$$

$$\langle W_{(5,3)} \rangle = -A^{20} + 5A^{16} - 10A^{12} + 15A^8 - 19A^4 + 21 - 19A^{-4} + 15A^{-8} - 10A^{-12} + 5A^{-16} - A^{-20} \quad (5.8)$$

$$\langle W_{(7,3)} \rangle = -A^{28} + 7A^{24} - 21A^{20} + 42A^{16} - 70A^{12} + 98A^8 - 118A^4 + 127 - 118A^{-4} + 98A^{-8} - 70A^{-12} + 42A^{-16} - 21A^{-20} + 7A^{-24} - A^{-28} \quad (5.9)$$

We can make a few observations about the behavior of the $(m, 3)$ weaving knot polynomial. Notice that the first few polynomials of the $(m, 3)$ weaving knot have symmetric terms with respect to the exponents. In other words, if we have $-A^j$ as a term in our polynomial, $-A^{-j}$ also turns up. Furthermore, the terms of each polynomial alternates signs with the first terms of the $(m, 3)$ polynomial as $(-1)^m A^{4m}$.

Curiously, the polynomial for the $(2, 3)$ weaving knot (Equation 5.6) is actually the same polynomial as the figure eight knot [1]. Recall that the $(3, 2)$ weaving knot is the same as the $(3, 2)$ torus knot, which is the trefoil. The polynomial for the trefoil, as we've calculated earlier is $A^4 + A^{12} - A^{16}$, which is not the same as the polynomial for the $(2, 3)$ weaving knot. Thus, since the Jones polynomial is a knot invariant, we know then that unlike the torus knots, the (m, n) weaving knot is not necessarily the same as the (n, m) weaving knot.

Chapter 6

Future Directions

We can apply our technique to any (m, n) weaving knot or (m, n) torus knot with n fixed. Furthermore, our technique extends easily to any knot that has a repeated pattern in its braid representation form. For example, we can easily iterate the polynomial for a knot with the braid representation of $(\sigma_1\sigma_3\sigma_2^{-1})^n$ where each twist of $\sigma_1\sigma_3\sigma_2^{-1}$ would be repeated n times.

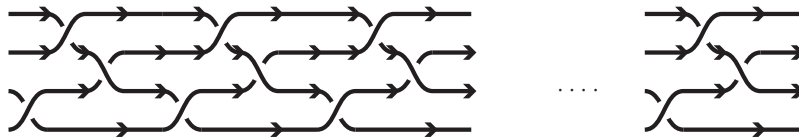


Figure 6.1: braid representation of $(\sigma_1\sigma_3\sigma_2^{-1})^n$

Using our technique, we would proceed by decomposing each crossing of $\sigma_1\sigma_3\sigma_2^{-1}$, explore the Temperley-Lieb Algebra relations on 4 strands, reattach the ends for our closing formulas, and finally calculate the writhe of the knot by calculating the writhe of each twist of $\sigma_1\sigma_3\sigma_2^{-1}$ then multiplying by n . In our case, the writhe of a single twist is -1. For a knot with n such twists in its braid representation, the writhe would be $-n$.

Future directions that would be worth exploring include perhaps finding a closed formula for polynomial of the $(m, 3)$ weaving knot, which in its unclosed braid representation form is the true braid. While we could inductively find the polynomial for an $(m, 3)$ weaving knot for a fixed m , a closed form is not apparently obvious through our technique. Another direction to explore would be to find a close form for the polynomial of any (m, n) weaving knot.

It would also be worthwhile to look into other families of knots that are

repeated by a pattern in its braid representation. Using our technique, we may be able to find a closed form for the polynomial of those families of knots.

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