MATH234
Solutions to Exam 1
June 29, 2011

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1. $r(t) = (2 + 3\cos t)\vec{i} + (5\sin t)\vec{j} + (4\cos t)\vec{k}$.

   (a)

$$\begin{aligned} v(t) &= -3(\sin t)\vec{i} + 5(\cos t)\vec{j} - 4(\sin t)\vec{k} \\ |v(t)| &= \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t} \\ &= \sqrt{25\sin^2 t + 25\cos^2 t} \\ &= 5 \end{aligned}$$

   (b) The particle is at point (2,5,0) when $r(t) = \langle 2, 5, 0\rangle$, which occurs when $t = \pi/2$. The direction of the tangent line at (2,5,0) is $v(\pi/2) = -3\vec{i} + 0\vec{j} - 4\vec{k}$.
   The parametric equations of the line passing through (2,5,0) moving in the direction $-3\vec{i} + 0\vec{j} - 4\vec{k}$ are:

$$x(t) = 2 - 3t, y(t) = 5, z(t) = -4t$$

   (c)

$$\begin{aligned} T &= v(t)/|v(t)| \\ &= (-3\sin t\vec{i} + 5\cos t\vec{j} - 4\sin t\vec{k})/5 \\ &= -\frac{3}{5}\sin t\vec{i} + \cos t\vec{j} - \frac{4}{5}\sin t\vec{k} \end{aligned}$$

   (d)

$$\begin{aligned} \kappa &= \frac{1}{|v(t)|}\left|\frac{dT}{dt}\right| \\ &= \frac{1}{5}\left|-\frac{3}{5}(\cos t)\vec{i} - (\sin t)\vec{j} - \frac{4}{5}(\cos t)\vec{k}\right| \\ &= \frac{1}{5}\sqrt{\frac{9}{25}\cos^2 t + \sin^2 t + \frac{16}{25}\cos^2 t} \\ &= \frac{1}{5} \end{aligned}$$

   (e)

$$\begin{aligned} N &= \frac{dT}{dt}\Big/\left|\frac{dT}{dt}\right| \\ &= \frac{dT}{dt} \text{ from previous computation} \\ &= -\frac{3}{5}(\cos t)\vec{i} - (\sin t)\vec{j} - \frac{4}{5}(\cos t)\vec{k} \end{aligned}$$

(f) To show that C is centered at (2,1,0), it suffices to show that every point on the circle is equidistant from (2,1,0) and that C is contained in a plane passing through (2,1,0). We can do the first part by showing that $|r - \langle 2,1,0\rangle|$ is constant.

$$\begin{aligned}|r - \langle 2,1,0\rangle| &= |3(\cos t)\vec{i} + 5(\sin t)\vec{j} + 4(\cos t)\vec{k} \\ &= \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t} = 5\end{aligned}$$

The second part is shown in part (c).

(g) In the previous computation, we showed that every point on C was 5 units from the center of the C, so the radius is 5.

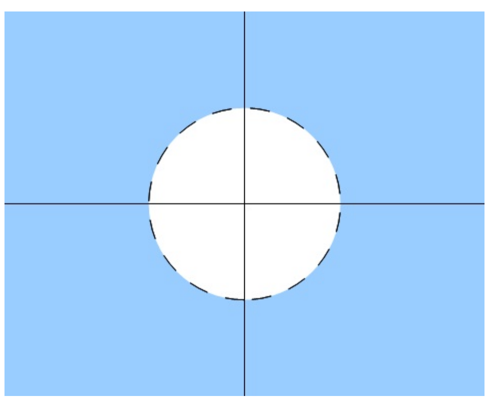
(h) Notice that $x = 2 + 3\cos t$, so $\cos t = (x-2)/3$. Then $z = 4\cos t = \frac{4}{3}(x-2) = \frac{4}{3}x - \frac{8}{3}$. This plane passes through $(2,1,0)$ and contains C.

2. $f(x,y) = \ln(x^2 + y^2 - 1)$

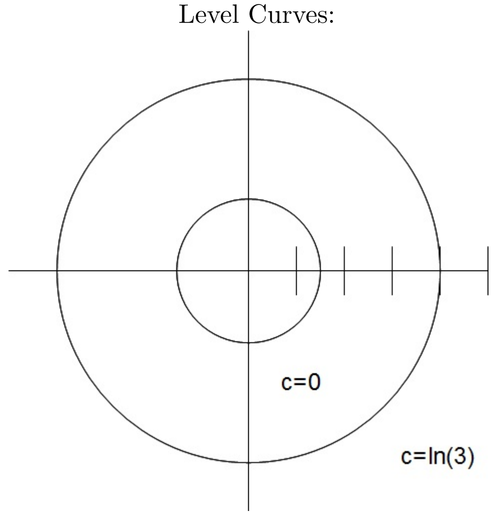
(a) $Dom(f)$ contains all points (x,y) such that $x^2 + y^2 - 1 > 0$, or $Dom(f) = \{(x,y)|x^2 + y^2 > 1\}$

(b) The level curve at $f(x,y) = 0$ is given by the equation $x^2 + y^2 = 2$. The level curve at $f(x,y) = \ln 3$ is given by the equation $x^2 + y^2 = 4$.

Domain:



Level Curves:



(c) The boundary points of the domain are (x,y) such that $x^2 + y^2 = 1$. $Dom(f)$ contains none of these boundary points, so it is open and not closed.

(d) $Dom(f)$ is not bounded since it cannot be contained in a disk of any radius.

3. $f(x,y) = x^2 - 4y^2 + 2xy - 3x + 5y - 1$

(a) The first-order partial derivatives are $f_x(x,y) = 2x + 2y - 3$ and $f_y(x,y) = -8y + 2x + 5$, so the second-order partial derivatives are:

$$\begin{aligned}f_{xx}(x,y) &= 2 \\ f_{xy}(x,y) &= 2 \\ f_{yx}(x,y) &= 2 \\ f_{yy}(x,y) &= -8\end{aligned}$$

(b)

$$\nabla f(x,y) = \langle 2x+2y-3, -8y+2x+5\rangle$$
$$\nabla f(-1,2) = \langle -1, -13\rangle$$

The maximum possible directional derivative at (-1,2) is $|\nabla f(-1,2)| = \sqrt{1+169} = \sqrt{170}$

4. $f(x,y) = \frac{x^2+y^2}{x^2-y^2}$

   (a) No, the limit does not exist. If you approach $(0,0)$ along the line $y=0$, then:

   $$f(x,0) = \frac{x^2}{x^2} = 1$$

   If you approach $(0,0)$ along the line $x=0$, then:

   $$f(0,y) = \frac{y^2}{-y^2} = -1$$

   Hence, there are two paths which produce two different limits, so the limit does not exist.

   (b) When $y \neq \pm x$, $f(x,y)$ is defined and is a rational function, so $f(x,y)$ is continuous. When $y = \pm x$, the function is not defined, so it is not continuous.

5. If $g_{xx}(x,y) = 0$, then $g_x(x,y)$ is a function of y, so we can write $g_x(x,y) = f(y)$.

   Then we must have $g(x,y) = f(y)x + h(y)$ for some function $h(y)$.

   Evaluating at $(0,y)$, we get $g(0,y) = f(y)(0) + h(y) = h(y)$. Thus, by assumption, $h(y) = \sin y$.

   Similarly, $g(1,y) = f(y)(1) + h(y) = f(y) + h(y)$. Thus, by assumption, $f(y) + h(y) = \cos y$. Since $h(y) = \sin y$, then $f(y) = \cos y - \sin y$.

   Therefore $g(x,y) = (\cos y - \sin y)x + \sin y$.