MATH 234

Solutions to Exam 2

1.

a. To show P is on the surface S, we must show that it satisfies the equation

$$x^2+y^2+z^2=2(x+y+z)+1.$$

$$(-1)^2+(1)^2+1^2=3=2(-1+1+1)+1$$

b. Let $f(x,y,z)=x^2+y^2+z^2-2(x+y+z)-1$. Then the surface S is the level surface $f(x,y,z)=0$. Thus, the plane tangent to the surface S at P is given by the equation:

$$f_x(-1,1,1)(x+1)+f_y(-1,1,1)(y-1)+f_z(-1,1,1)(z-1)=0$$

The partial derivatives are:

$$\begin{array}{lll} f_x=2x-2, & f_y=2y-2, & f_z=2z-2 \\ f_x(-1,1,1)=-4, & f_y(-1,1,1)=0, & f_z(-1,1,1)=0 \end{array}$$

Thus, the equation of the plane is:

$$\begin{aligned} -4(x+1)&=0 \\ -4x&=4 \\ x&=-1 \end{aligned}$$

c. A normal vector to the surface is $\langle -4,0,0\rangle$ and $(-1,1,1)$ is a point on the line. This gives us parametric equations $x=-1-4t$, $y=1$, $z=1$ for the line normal to the surface at P.

2.

a. The quadratic approximation of f is equal to the sum of quadratic approximation of $\cos x$ and the quadratic approximation of $\sin y$.

The quadratic approximation of $\cos x$ is $1-\frac{x^2}{2}$. The quadratic approximation of $\sin y$ is y. Thus, $Q=1+y-\frac{x^2}{2}$.

b. There are numerous ways to find a bound on the error. One way is to bound the error term, as follows:

$$\frac{1}{3!}\left|x^3 f_{xxx}+3x^2y\, f_{xxy}+3x\, y^2 f_{xyy}+y^3 f_{yyy}\right|_{(cx,cy)}$$

We have $f_{xxx}=\sin x$, $f_{xxy}=0$, $f_{xyy}=0$, $f_{yyy}=-\cos y$. Thus,

$$\begin{aligned} \text{Error} &= \frac{1}{6}\left|x^3\sin(c\,x)-y^3\cos(c\,y)\right| \\ &\le \frac{1}{6}\left(|x|^3|\sin(.1)|+|y|^3|\cos(0)|\right) \\ &\le \frac{1}{6}\left((.1)^3(.1)+(.2)^3(1)\right) \\ &= .00135 \end{aligned}$$

3. The average value is:

$$\frac{1}{\text{Area}}\iint_R f(x,y)\,dA$$

The area of integration is a triangle with base 1 and height 1, so it has area 1/2. It has bounds $0 \leq 1 \leq y$ and $0 \leq x \leq y$.

$$\begin{aligned}\text{Average Value} &= \frac{1}{\text{Area}} \int\int_R f(x,y)dA \\ &= \frac{1}{1/2} \int_0^1 \int_0^y e^{-y^2} dx\,dy \\ &= 2 \int_0^1 \int_0^y e^{-y^2} dx\,dy \\ &= 2 \int_0^1 y e^{-y^2} dy \\ &= -\int_0^1 -2y e^{-y^2} dy \\ &= -\left[e^{-y^2}\right]_0^1 \\ &= -(e^{-1} - e^0) \\ &= 1 - e^{-1}\end{aligned}$$

4.

a. The first moment around the x-axis is given by the integral:

$$M_x = \int\int_R y\delta dA$$

The circular plate is the circle of radius 2 centered at (2, 0). The bounds of the region are $0 \leq x \leq 4$ and $-\sqrt{4x - x^2} \leq y \leq \sqrt{4x - x^2}$. Thus,

$$M_x = \int_0^4 \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} y(4-x) dy\,dx$$

b. The first moment around the y-axis is given by the integral:

$$M_y = \int\int_R x\delta dA$$

In polar coordinates, the plate is given by $r^2 \leq 4r\cos\theta$, or $r \leq 4\cos\theta$, and $-\pi/2 \leq \theta \leq \pi/2$. Thus,

$$M_y = \int_{-\pi/2}^{\pi/2} \int_0^{4\cos\theta} (r\cos\theta)(4 - r\cos\theta) r\,dr\,d\theta$$

c. The plate is symmetric about the y-axis and the density δ is also symmetric about the y-axis ($\delta(x,y) = \delta(x,-y)$). Thus, $\overline{y} = 0$.

5. The two surfaces intersect when $x^2 + y^2 = 4 - x^2 - y^2$, so when $x^2 + y^2 = 2$. Thus, the region of integration is $-\sqrt{2} \leq x \leq \sqrt{2}, -\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2}$, and $x^2 + y^2 \leq z \leq 4 - x^2 - y^2$. In cylindrical coordinates, $0 \leq r \leq \sqrt{2}, 0 \leq \theta \leq 2\pi, r^2 \leq z \leq 4 - r^2$.
Thus, the volume is computed with the following integral:

$$\begin{aligned}\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} r\,dz\,dr\,d\theta &= \int_0^{2\pi} \int_0^{\sqrt{2}} [r\,z]_{r^2}^{4-r^2} dr\,d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} 4r - 2r^3 dr\,d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{2}\right]_0^{\sqrt{2}} d\theta \\ &= \int_0^{2\pi} 2d\theta \\ &= [2\theta]_0^{2\pi} \\ &= 4\pi\end{aligned}$$

6. The moment of inertia about the z-axis is given by the following integral:

$$I_z = \iiint (x^2 + y^2) dV$$

The region of integration is given in spherical coordinates by $a \leq \rho \leq b, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$.

$$\begin{aligned}
I_z &= \int_0^{2\pi} \int_0^{\pi} \int_a^b ((\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2) \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_a^b (\rho^2 \sin^2\phi) \rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_a^b \rho^4 \sin^3\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \sin^3 \left[\frac{\rho^5}{5}\right]_a^b d\phi d\theta \\
&= \frac{b^5 - a^5}{5} \int_0^{2\pi} \int_0^{\pi} \sin^3 d\phi d\theta \\
&= \frac{b^5 - a^5}{5} \int_0^{2\pi} \int_0^{\pi} \sin\phi (1 - \cos^2\phi) d\phi d\theta \\
&= \frac{b^5 - a^5}{5} \int_0^{2\pi} \int_0^{\pi} \sin\phi - \sin\phi \cos^2\phi d\phi d\theta \\
&= \frac{b^5 - a^5}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} d\theta \\
&= \frac{b^5 - a^5}{5} \int_0^{2\pi} \left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] d\theta \\
&= \frac{b^5 - a^5}{5} \frac{4}{3} \int_0^{2\pi} d\theta \\
&= \frac{b^5 - a^5}{5} \frac{4}{3} 2\pi \\
&= \frac{8\pi(b^5 - a^5)}{15}
\end{aligned}$$

7.

a. The critical points are the points at which $f_x = 0$ and $f_y = 0$. $f_x = -2(x + y)$ and $f_y = -2(x + y)$. Both f_x and f_y are 0 When $-2x - 2y = 0$, so when $y = -x$. All points $(x, -x)$ are critical points.

Note that $f(x, y) \leq 5$ for all points (x, y), and for all x,

$$f(x, -x) = 5 - (x - x)^2 = 5.$$

Therefore $(x, -x)$ is an absolute maximum for all x.

b. First, note that $g_x = 2x y^3$ and $g_y = 3x^2 y^2$, so $g_x(0, 0) = 0$ and $g_y(0, 0) = 0$. Thus, $(0, 0)$ is a critical point.

At (0,0), $g(0, 0) = 0$ If you move any distance from (0,0) into the first quadrant, $g(x, y)$ becomes positive. Thus $(0, 0)$ cannot be a local maximum. Similarly, if you move any distance from (0,0) into the 4th quadrant, $g(x, y)$ becomes negative. Thus (0, 0) cannot be a local minimum. A critical point is always a local maximum, local minimum, or a saddle point, so $(0, 0)$ is a saddle point.