# MATH234
## Solutions to Exam 3
## August 4, 2011

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1. (a) The mass of the wire is computed with the integral:

$$\oint_C \delta(x,y)ds$$

where $C$ is parameterized as $r(t) = \langle 2\cos t, 2\sin t\rangle$ with $0 \le t \le 2\pi$. Then $r'(t) = \langle -2\sin t, 2\cos t\rangle$, so $|r'(t)| = 2$. Then:

$$\begin{aligned}\oint_C \delta(x,y)ds &= \int_0^{2\pi} \delta(2\cos t, 2\sin t)|r'(t)|dt \\ &= \int_0^{2\pi} (2-2\cos t)2dt \\ &= 4\int_0^{2\pi} 1-\cos t dt \\ &= 4\left[t-\sin t\right]_0^{2\pi} \\ &= 8\pi\end{aligned}$$

(b) The moment of inertia about the origin is computed with the integral:

$$I_0 = \oint_C (x^2+y^2)\delta(x,y)ds$$

Since $C$ is the path along $x^2+y^2=4$, then that integral will be equal to:

$$\oint_C 4\delta(x,y)ds = 4\oint_C \delta(x,y)ds = 4(\text{Mass})$$

Thus, the moment of inertia is $4(8\pi) = 32\pi$.

2. (a) The work done is computed with the integral:

$$\int_C \vec{F}\cdot d\vec{r}$$

where $\vec{r} = \langle t, 0, -2t\rangle$ for $0 \le t \le 1$. Thus $d\vec{r} = \langle 1, 0, -2\rangle$, so we have

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 \vec{F}(r(t))\cdot\langle 1,0,-2\rangle dt \\ &= \int_0^1 \langle 0+3(-2t), 0, 3(t)+0\rangle\cdot\langle 1,0,-2\rangle dt \\ &= \int_0^1 -12tdt \\ &= \left[-4t^2\right]_0^1 \\ &= -6\end{aligned}$$

(b) Yes, because $\vec{F}$ is not conservative. If $\vec{F}$ were conservative, then if we write $\vec{F} = M\vec{i}+N\vec{j}+P\vec{k}$, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. In this case, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0$. They are not equal, so $\vec{F}$ is not conservative.

3. (a) By Green's Theorem, since the path is clockwise

$$\begin{aligned}
\oint_c (ax+by)dx + (cx+dy)dy &= -\int\int_R \left( \frac{\partial(cx+dy)}{\partial x} - \frac{\partial(ax+by)}{\partial y} \right) dxdy \\
&= -\int\int_R (c-b)dxdy \\
&= -(c-b)\int\int_R dxdy \\
&= -(c-b) \text{ Area of } R
\end{aligned}$$

where $R$ is the region contained in the circle $c$. $R$ is a circle of radius $r$, so it has area $\pi r^2$. Thus, the integral equals $-(c-b)\pi r^2$, or $(b-c)\pi r^2$.

Note that the integral does not depend on the location of the center of $c$ since the resulting integral only required information about the area of the circle.

(b) Yes, if the path were counterclockwise, by Green's Theorem, the double integral above would be positive instead of negative. The resulting integral would be $(c-b)\pi r^2$.

4. (a) One valid parametrization is $r(u,v) = \langle u\cos v, u\sin v, 1-u\rangle$ with $0 \le u \le 1$ and $0 \le v \le \frac{\pi}{2}$.

(b) We compute $d\sigma$ using the parameterizations from part (a).

$$\begin{aligned}
r_u &= \langle \cos v, \sin v, -1\rangle \\
r_v &= \langle -u\sin v, u\cos v, 0\rangle \\
r_u \times r_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & -1 \\ -u\sin v, u\cos v, 0 & & \end{vmatrix} \\
&= \langle u\cos v, u\sin v, u\cos^2 v + u\sin^2 v\rangle \\
&= \langle u\cos v, u\sin v, u\rangle \\
|r_u \times r_v| &= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = u\sqrt{2} \\
\int\int_S \cos(x^2+y^2)d\sigma &= \int_0^1\int_0^{\pi/2} \cos(u^2\cos^2 v + u^2\sin^2 v)u\sqrt{2}dvdu \\
&= \sqrt{2}\int_0^1\int_0^{\pi/2} u\cos(u^2)dvdu \\
&= \frac{\sqrt{2}\pi}{2}\int_0^1 u\cos(u^2)du \\
&= \frac{\sqrt{2}\pi}{4}\left[\sin(u^2)\right]_0^1 \\
&= \frac{\sqrt{2}\pi}{4}\sin(1)
\end{aligned}$$

5. (a) One approach is to recognize that on the sphere of radius $a$, the function $f(x,y,z) = \sqrt{x^2+y^2+z^2} = a$. Thus,

$$\int\int_S f(x,y,z)d\sigma = a\int\int_S d\sigma = a(\text{Surface Area of the sphere})$$

The surface area of a sphere of radius $a$ is $4\pi a^2$, so the integral is $4\pi a^2$.

Another (longer) approach to this problem is to parameterize the surface and use the parameterizations to compute $d\sigma$.

Let $r(\varphi, \theta) = \langle a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi\rangle$ with $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi$. Then:

$$\begin{aligned} r_\varphi &= \langle a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi\rangle \\ r_\theta &= \langle -a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0\rangle \\ r\varphi \times r\theta &= \langle a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi\rangle \\ |r\varphi \times r\theta| &= a^2\sin\varphi \end{aligned}$$

On the surface, $f(x, y, z,) = \sqrt{x^2+y^2+z^2} = a$. Thus,

$$\begin{aligned} \int\int_S f(x,y,z)d\sigma &= \int_0^\pi \int_0^{2\pi} a(a^2\sin\varphi)d\theta d\varphi \\ &= a^3 2\pi \int_0^\pi \sin\varphi d\varphi \\ &= a^3 2\pi \left[-\cos\varphi\right]_0^\pi \\ &= 4\pi a^3 \end{aligned}$$

(b) The circulation around any closed curve in a conservative vector field is 0. $\nabla f$ is conservative, since $f$ is a potential function for $\nabla f$. Thus, the circulation is 0.

Another approach:

If a simple closed path $C$ avoids the origin, we can apply Stokes' Theorem to get that:

$$\text{Circulation} = \oint_C \nabla f \cdot d\vec{r} = \int\int_S \nabla \times \nabla f \cdot \vec{n} d\sigma$$

For every function $f$, $\nabla \times \vec{\nabla f} = 0$. Thus, the circulation is zero.

We could not apply Stokes' Theorem if the curve passed through the origin, since $\nabla F$ is not defined at the origin.

6. (a) By the Divergence Theorem,

$$\int\int_S \vec{F}\cdot\vec{n}d\sigma = \int\int\int_D \nabla\cdot\vec{F}dV$$

For $\vec{F} = (x+2y+3z)\vec{i}$, $\nabla\cdot\vec{F} = 1$. Thus, the above integral reduces to:

$$\int\int\int_D dV$$

or, the volume of the pyramid. The volume of a pyramid is $\frac{1}{3}$(Base area) (Height). The base is a triangle with height 3 and length 6, so the base area is 9. The height is 2, so the volume is 6.

You can also compute the volume with a triple integral, like:

$$\int_0^2 \int_0^{3-3/2} \int_0^{6-2y-3z} dxdydz$$

That will also give you 6.

Note: If you wanted to compute this integral without the Divergence theorem, you must compute the flux over all four sides of the pyramid. Two of them have zero flux, but the top of the pyramid and the side in the yz-plane have non-zero flux.

(b) By the divergence theorem again,

$$\int\int_S (\nabla \times \vec{F}) \cdot \vec{n} d\sigma = \int\int\int_D \nabla \cdot (\nabla \times \vec{F}) dV$$

It is true in general that $\nabla \cdot \nabla \times \vec{F} = 0$ for any vector field $\vec{F}$, but you can also just check it for this vector field in particular.

$$\begin{aligned} \nabla \times \vec{F} &= \langle 0, 3, -2 \rangle \\ \nabla \cdot (\nabla \times \vec{F}) &= 0 \end{aligned}$$

Thus, the integral above becomes zero, so the flux is zero.