

MATH234

Solutions to Sample Exam 2

<today>

1.

- a. The level surface of $f(x, y, z)$ at P is

$$(x - 2y - 5)^2 + (x + y - 2)^2 + (x + z - 3)^2 = 13.$$

The tangent plane to the level surface at P has equation

$$f_x(1, -2, 0)(x - 1) + f_y(1, -2, 0)(y + 2) + f_z(1, -2, 0)(z - 0) = 0$$

Then $f_x = 2(x - 2y - 5) + 2(x + y - 2) + 2(x + z - 3)$, so $f_x(1, -2, 0) = -10$.

$f_y = -4(x - 2y - 5) + 2(x + y - 2)$, so $f_y(1, -2, 0) = -6$.

$f_z = 2(x + z - 3)$, so $f_z(1, -2, 0) = -4$.

Thus, the tangent plane has equation

$$\begin{aligned} -10(x - 1) - 6(y + 2) - 4z &= 0 \\ -10x - 6y - 4z &= 2 \end{aligned}$$

- b. For all (x, y, z) , $f(x, y, z) \geq 0$ since it is the sum of squares. Any point Q such that $f(Q) = 0$ is an absolute minimum.

$$f(x, y, z) = 0 \text{ when } x - 2y - 5 = 0, x + y - 2 = 0, \text{ and } x + z - 3 = 0.$$

If $x - 2y - 5 = 0$, then $x = 2y + 5$. By the second equation, then

$$2y + 5 + y - 2 = 0, \text{ so } y = -1. \text{ Thus } x = 3.$$

By the third equation, $3 + z - 3 = 0$, so $z = 0$, and the function has an absolute minimum. $Q = (3, -1, 0)$ is

2.

- a. No, it does not. If it did, then it would have a critical point, or a point where $f_x = 0$ and $f_y = 0$. Since $f_x = 3$ and $f_y = -4$, there are no critical points.

- b. By part (a), we know that any extrema must lie on the boundary, $x^2 + y^2 = 1$. Then $y = \pm \sqrt{1 - x^2}$.

If $y = \sqrt{1 - x^2}$, then $f(x, y) = 3x - 4\sqrt{1 - x^2} + 2$. That has extrema when $f' = 0$, so when $3 + 4x(1 - x^2)^{-1/2} = 0$. Solving for x , this occurs when $x = \pm \frac{3}{5}$. In both cases, $y = \frac{4}{5}$.

If $y = -\sqrt{1 - x^2}$, then $f(x, y) = 3x + 4\sqrt{1 - x^2} + 2$. That has extrema when $f' = 0$, so when $3 - 4x(1 - x^2)^{-1/2} = 0$. Solving for x , this also occurs when $x = \pm \frac{3}{5}$. In both cases, $y = -\frac{4}{5}$.

Thus, the critical points are $(3/5, 4/5)$, $(3/5, -4/5)$, $(-3/5, 4/5)$, $(-3/5, -4/5)$. By plugging these into the original equation, we see that the absolute maximum is at $(3/5, -4/5)$ and the absolute minimum is at $(-3/5, 4/5)$.

3.

- a. Using the Maclaurin expansion of e^t , we get that the Taylor series expansion for $e^{-(x^2+y^2)}$ about the origin is:

$$1 - (x^2 + y^2) + \frac{(x^2 + y^2)^2}{2} - \frac{(x^2 + y^2)^3}{6} + \dots$$

Thus, the cubic approximation is:

$$1 - (x^2 + y^2)$$

- b. Since the Taylor series for $f(x, y)$ is an alternating series, the error in the approximation $1 - (x^2 + y^2)$ is no greater than the absolute value of the next term in the series, $-\frac{(x^2 + y^2)^2}{2}$. Thus:

$$\begin{aligned} E &\leq \left| -\frac{(x^2 + y^2)^2}{2} \right| = \frac{1}{2} |x^4 + 2x^2y^2 + y^4| \\ &\leq \frac{1}{2} (|x|^4 + 2|x|^2|y|^2 + |y|^4) \\ &\leq \frac{1}{2} ((.1)^4 + 2(.1)^2(.1)^2 + (.1)^4) \\ &= \frac{1}{2} (4(.1)^4) = .0002 \end{aligned}$$

So the error is at most .0002.

4.

$$\begin{aligned} \int_0^1 \int_y^1 \sin\left(\frac{y}{x}\right) dx dy &= \int_0^1 \int_0^x \sin\left(\frac{y}{x}\right) dy dx \\ &= \int_0^1 x \int_0^x \frac{1}{x} \sin\left(\frac{y}{x}\right) dy dx \\ &= \int_0^1 x \left[-\cos\left(\frac{y}{x}\right) \right]_0^x dx \\ &= \int_0^1 x [-\cos(1) + \cos(0)] dx \\ &= (1 - \cos(1)) \int_0^1 x dx \\ &= (1 - \cos(1)) \frac{1}{2} \end{aligned}$$

5.

$$\begin{aligned} \int_0^2 \int_x^{\sqrt{8-x^2}} e^{1-x^2-y^2} dy dx &= \int_{\pi/4}^{\pi/2} \int_0^{2\sqrt{2}} e^{1-r^2} r dr d\theta \\ &= \frac{-1}{2} \int_{\pi/4}^{\pi/2} \int_0^{2\sqrt{2}} -2r e^{1-r^2} dr d\theta \\ &= \frac{-1}{2} \int_{\pi/4}^{\pi/2} \left[e^{1-r^2} \right]_0^{2\sqrt{2}} d\theta \\ &= \frac{-1}{2} \int_{\pi/4}^{\pi/2} e^{-7} - e^1 d\theta \\ &= \frac{e - e^{-7}}{2} \int_{\pi/4}^{\pi/2} d\theta \\ &= \frac{\pi(e - e^{-7})}{8} \end{aligned}$$

6. The half ball is described by the bounds $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$, and $0 \leq \rho \leq a$.

$$\begin{aligned} M &= \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \delta \rho^2 \sin \phi d\theta d\phi d\rho \\ &= \delta \left(\int_0^a \rho^2 d\rho \right) \left(\int_0^{\pi/2} \sin \phi d\phi \right) \left(\int_0^{2\pi} d\theta \right) \\ &= \delta \left(\frac{a^3}{3} \right) (1) (2\pi) \\ &= \frac{2\pi a^3}{3} \delta \\ M_{yz} &= \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \delta (\rho \sin \phi \cos \theta) \rho^2 \sin \phi d\theta d\phi d\rho \\ &= \delta \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \rho^3 \sin^2 \phi \cos \theta \end{aligned}$$

$$\begin{aligned}
&= \delta \int_0^a \int_0^{\pi/2} [\rho^3 \sin^2 \phi \sin \theta]_0^{2\pi} d\phi d\rho \\
&= \delta \int_0^a \int_0^{\pi/2} 0 d\phi d\rho \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
M_{xz} &= \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \delta(\rho \sin \phi \sin \theta) \rho^2 \sin \phi d\theta d\phi d\rho \\
&= 0 \\
M_{xy} &= \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \delta(\rho \cos \phi) \rho^2 \sin \phi d\theta d\phi d\rho \\
&= \delta \int_0^a \int_0^{\pi/2} \int_0^{2\pi} \rho^3 \sin \phi \cos \phi d\theta d\phi d\rho \\
&= \delta 2\pi \int_0^a \int_0^{\pi/2} \rho^3 \sin \phi \cos \phi d\phi d\rho \\
&= \delta 2\pi \int_0^a \rho^3 \left[-\frac{1}{2} \cos^2 \phi \right]_0^{\pi/2} d\rho \\
&= \delta \pi \int_0^a \rho^3 d\rho \\
&= \delta \pi \left[\frac{\rho^4}{4} \right]_0^a \\
&= \frac{\delta \pi a^4}{4}
\end{aligned}$$

So the centroid is $(M_{yz}/M, M_{xz}/M, M_{xy}/M) = (0, 0, \frac{3a}{8})$

7. The paraboloid and the cone intersect at $1 - x^2 - y^2 = 1 - \sqrt{x^2 + y^2}$, which is equal to $\sqrt{x^2 + y^2}(1 - \sqrt{x^2 + y^2}) = 0$. Thus, they intersect at the vertex $(0,0,1)$ and at the curve $x^2 + y^2 = 1$ in the xy -plane.

All points within the region have $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This means that $0 \leq \sqrt{x^2 + y^2} \leq 1$ and $x^2 + y^2 \leq \sqrt{x^2 + y^2}$. Therefore $1 - x^2 - y^2 \geq 1 - \sqrt{x^2 + y^2}$, so the paraboloid is the upper surface and the cone is the lower surface.

Thus, our region of integration is bounded by $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r = 1 - \sqrt{x^2 + y^2} \leq z \leq 1 - x^2 - y^2 = 1 - r^2$.

$$\begin{aligned}
\int_0^{2\pi} \int_0^1 \int_{1-r}^{1-r^2} r dz dr d\theta &= \int_0^{2\pi} \int_0^1 r(1 - r^2 - 1 + r) dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (-r^3 + r^2) dr d\theta \\
&= \int_0^{2\pi} \left[-\frac{r^4}{4} + \frac{r^3}{3} \right]_0^1 d\theta \\
&= \int_0^{2\pi} \frac{1}{12} d\theta \\
&= \frac{1}{12} 2\pi = \frac{\pi}{6}
\end{aligned}$$

So the volume is $\pi/6$.