a) show \( y_1, y_2 \) are linearly independent on any interval \( I \) of \( \mathbb{R} \).

Recall two functions are linearly independent on \( I \) if neither of them is a constant multiple of the other.

1) Suppose \( y_1 = Cy_2 \), i.e., \( \sin x^2 = C \cos x^2 \).

Then \( \tan x^2 = C \), but \( \tan x^2 \) is not a constant valued function

Thus this case can't happen.

2) Also if \( y_2 = Cy_1 \), get \( \cos x^2 = C \sin x^2 \), \( \cot x^2 = C \).

But \( \cot x^2 \) is not a constant valued function, so this case can't happen.

Hence \( y_1 \) and \( y_2 \) are linearly independent on any interval of \( \mathbb{R} \).

b) Wronskian vanishes at \( x = 0 \).

\[
W = \begin{vmatrix}
  y_1 & y_2 \\
  y_1' & y_2'
\end{vmatrix}
= \begin{vmatrix}
  \sin x^2 & \cos x^2 \\
  2x \cos x^2 & -2x \sin x^2
\end{vmatrix}
= -2x \sin^2(x^2) - 2x \cos^2(x^2)
= -2x (\sin^2(x^2) + \cos^2(x^2)) = -2x = 0 \text{ if } x = 0.
\]

c) Why does this imply that there is no DEQ of the form \( y'' + p(x)y' + q(x)y = 0 \)

with both \( p \) and \( q \) continuous everywhere, having both \( y_1 \) and \( y_2 \) as solutions?

This means \( I = \mathbb{R} \).

By thm 3, suppose \( y_1 \) and \( y_2 \) are two solutions of \( y'' + p(x)y' + q(x)y = 0 \)

on an open interval \( I \) on which \( p \) and \( q \) are continuous.

Since we showed in part a) that \( y_1, y_2 \) are linearly independent, by thm 3, then \( W(y_1, y_2) \neq 0 \) at each point of \( \mathbb{R} \).

This is a contradiction to part b), so \( y_1, y_2 \) can't be solutions of the form \( y'' + p(x)y' + q(x)y = 0 \).