$\boxed{3.4.3}$ We are told that $F(x)$ has a jump at $x = x_0$, and that $F(x)$ is given by a cosine series

$$\Longrightarrow \quad F(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

with $a_0 = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx = \frac{1}{L}\int_0^L F(x)\,dx \qquad n=0$

$$a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L F(x)\cos\frac{n\pi x}{L}\,dx \qquad n \neq 0$$

We are also going to represent

$$\frac{dF(x)}{dx} \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$n \neq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \frac{dF(x)}{dx}\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L \frac{dF(x)}{dx}\sin\frac{n\pi x}{L}\,dx$$

Take $x_0 > 0$ so that we can work in $[0, L]$

{think about the other cases...}

How do the a_n's and b_n's relate to each other?

Note that if we integrate by parts

$$b_n = \frac{2}{L}\int_0^L \frac{dF(x)}{dx} \sin\frac{n\pi x}{L}\, dx \qquad u = \sin\frac{n\pi x}{L} \qquad dv = \frac{dF}{dx}dx$$

$$du = \frac{n\pi}{L}\cos\frac{n\pi x}{L}dx \qquad v = F$$

we will generate terms that look like

$$\int_0^L F(x)\cos\frac{n\pi x}{L}\,dx$$

which is basically a_n, $n \neq 0$ so this approach looks promising

Now do it carefully, remembering that $F(x)$ has a jump at $x = x_0$

$$b_n = \frac{2}{L}\int_0^L \frac{df(x)}{dx} \sin\frac{n\pi x}{L}\, dx \qquad n \neq 0$$

$$b_n = \frac{2}{L}\left\{ f(x)\sin\frac{n\pi x}{L}\Big|_0^{x_0^-} + f(x)\sin\frac{n\pi x}{L}\Big|_{x_0^+}^{L} \right.$$

$$\left. - \int_0^L \frac{n\pi}{L} f(x) \cos\frac{n\pi x}{L}\, dx \right\}$$

$$b_n = \frac{2}{L}\left\{ f(x_0^-)\sin\frac{n\pi x_0^-}{L} - \cancel{f(0)\sin(0)}^{\,0} \right.$$

$$+ f(L)\sin n\pi - f(x_0^+)\sin\frac{n\pi x_0^+}{L}$$

$$\left. - \frac{n\pi}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx \right\}$$

$$b_n = \frac{2}{L}\left[f(x_0^-) - f(x_0^+)\right]\sin\frac{n\pi x_0}{L} \qquad \left\{ \text{since } \sin\frac{n\pi x}{L} \text{ continuous at } x = x_0 \right\}$$

$$- \frac{n\pi}{L}\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$$

$$b_n = \frac{2}{L}(\alpha - \beta)\sin\frac{n\pi x_0}{L} - \frac{n\pi}{L} a_n \qquad n \neq 0$$

or $$a_n = -\frac{L}{n\pi} b_n - \frac{2}{n\pi}(\beta - \alpha)\sin\frac{n\pi x_0}{L}$$

What is this?

$$f(x) \sim -\sum_{n=1}^{\infty}\left[\frac{L}{n\pi} b_n + \frac{2}{n\pi}(\beta-\alpha)\sin\frac{n\pi x_0}{L}\right]\cos\frac{n\pi x}{L}$$

$$\sim \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Shows explicitly the expected $\frac{1}{n}$ slow convergence due to the jump:

the term $\left(-\frac{2}{n\pi}(\beta-\alpha)\sin\frac{n\pi x_0}{L}\right)\cos\frac{n\pi x}{L}$