"Every Day" Example: 1st-order linear ODES, systems, connection to PDEs!

Consider the temperature in a room as a function of time, with temp. outside the room everywhere $T_e$.

$$\frac{dT(t)}{dt} = -k(T(t) - T_e) \quad k > 0 \text{ constant}$$

Make sense?

* When $T(t) = T_e \Rightarrow \frac{dT}{dt} = 0$

  steady-state solution

* When $T(t) < T_e$, $T(t) - T_e < 0$,

  $-k(T - T_e) > 0$, $\frac{dT}{dt} > 0$

  temp will rise to $T_e$

* When $T(t) > T_e$ ..
This assumes a heat bath outside a $T_e$ and no heating/air conditioning in the room (we turned that off)

* What is $k$? By dimensional analysis

$$[k] = \left[ \frac{1}{\text{time}} \right] \text{ e.g. } \left[ \frac{1}{5} \right]$$

$k$ depends on the material of the walls/windows

* e.g. $k_s =$ single pain windows
  $k_d =$ double pain windows

$$k_d < k_s$$

* Note that we can write this equation in homogeneous form assuming $T_e$ constant

$$\frac{d}{dt} (T(t) - T_e) = -k (T(t) - T_e)$$

$$\frac{dy(t)}{dt} = -ky(t) \quad y(t) = T(t) - T_e$$
Then easy to see $y(t) = C \exp(-kt)$

$C$ arbitrary, check by plugging back in:

$T(t) - T_e = C \exp(-kt)$

* What is $C$? Need one more piece of info.

$T(t=0) = T_0 \Rightarrow C = T_0 - T_e$

$\Rightarrow T(t) = T_e + (T_0 - T_e) \exp(-kt)$ is

the unique solution to the

IVP = ODE + initial condition

* For more general case with heat/air conditioning

$\frac{dT(t)}{dt} = -k(T(t) - T_e) + R(t)$

$k > 0$, $R(t)$ given, $T(t=t_0) = T_0$

1st-order, linear, non-homogeneous ODE
Systems

Think about multiple connected rooms

\[ \text{Window} \ Ö Ö \text{Window} \]

- Room 1 loses/gains heat through the window to the outside, through the wall to Room 2

- Room 2 loses/gains heat through the walls to Rooms 1 and 3

Room 3...

\[ \rightarrow \text{coupled, 1st-order, linear ODEs} \]
\[
\text{Rm 1} \quad \frac{dT_1}{dt} = -k_{1e}(T_1 - T_e) - k_{12}(T_1 - T_2) + R_1
\]
\[
\text{Rm 2} \quad \frac{dT_2}{dt} = -k_{21}(T_2 - T_1) - k_{23}(T_2 - T_3) + R_2
\]
\[
\text{Rm 3} \quad \frac{dT_3}{dt} = -k_{32}(T_3 - T_2) - k_{3e}(T_3 - T_e) + R_3
\]

all \( k \)'s > 0 constant \( T_e > 0 \) constant

\( k_{12} = k_{21}, \ k_{23} = k_{32} \) by symmetry

\( T_1(t), \ T_2(t), \ T_3(t), \ R_1(t), \ R_2(t), \ R_3(t) \)

unknowns
given

Check the signs!

What if we had 100 rooms?

The steady state solution:

\[
\frac{dT_1(t)}{dt} = \frac{dT_2(t)}{dt} = \frac{dT_3(t)}{dt} = \cdots = 0
\]
A system of linear algebraic equations, so need LA.

1st find the steady-state, then the transient approach to steady state.

* Finally, what is the connection to PDE heat equation \( \frac{dT(x,t)}{dt} = \nu \frac{d^2T(x,t)}{dx^2} \)?

\[
\begin{array}{c|c|c|c}
\hline
\hline
 & T_{j-1} & T_j & T_{j+1} \\
\hline
\hline
\end{array}
\]

\[
\frac{dT_j}{dt} = -k(T_j - T_{j-1}) - k(T_j - T_{j+1})
\]

where there is only one k because we assume a homogeneous material [think now a pipe] or a rod.

different use of the word homogeneous!
This is the difference of 2 differences, or a 2nd-derivative in disguise!

\[
\frac{dT_i}{dt} = k(\Delta x)^2 \left\{ \frac{1}{\Delta x} \sum \left( \frac{T_{i+1}-T_i}{\Delta x} \right) \right\} - \left( \frac{T_i-T_{i-1}}{\Delta x} \right)^2
\]

Can we take the limit \( \Delta x \to 0 \)?

Must mean \( k \propto \frac{1}{(\Delta x)^2} \).

This is true in the continuous problem!
\[
\begin{align*}
\text{time rate of change of energy in sector } j &= \text{ energy in sector } j \quad \text{unit time} \\
(CADx) \frac{dT_j^i}{dt} &= -\alpha A \frac{\sum (T_j^e - T_{j+i}) + (T_j^w - T_{j-i})}{\Delta x} \\
&\uparrow \quad \text{volume of sector } j \quad \text{inversely proportional to thickness of boundary element} \\
&\quad \text{inversely proportional to thickness of boundary element} \\
\frac{dT_j^i}{dt} &= \frac{\alpha}{C(\Delta x)^2} \frac{(\Delta x)^2}{(\Delta x)^2} \sum (T_{j+i} - T_j^i) - (T_j^i - T_{j-i}) \\
\text{New limit as } \Delta x \to 0 \quad \Rightarrow \\
\frac{dT(x,t)}{dt} &= \nu \frac{\partial^2 T(x,t)}{\partial x^2} \\
\nu &= \frac{\alpha}{C}
\end{align*}
\]

in the continuous limit where the thickness of the section = thickness of the boundary