Math 322 Lecture 1 9/7/16

Reading: Haberman 1.1, 1.2, 1.5

Introduction to PDEs (partial differential equations)

3 fundamental PDEs:

* Heat Equation (conduction)

$$\frac{\partial T(\underline{x},t)}{\partial t} = k \nabla^2 T(\underline{x},t) + Q(\underline{x},t)$$

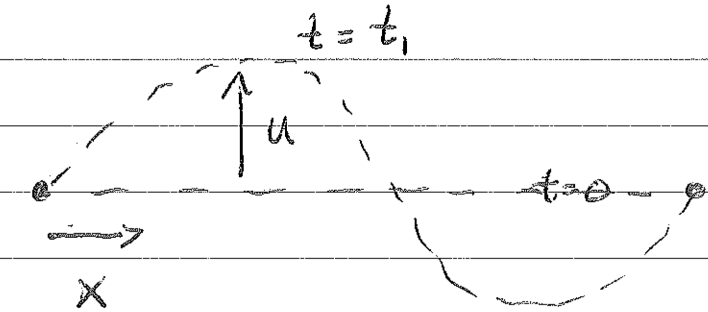
* Poisson's Equation (steady state heat eqn)

$$0 = k \nabla^2 T(\underline{x},t) + Q(\underline{x},t)$$

* The Wave Equation

$$\frac{\partial^2 u(\underline{x},t)}{\partial t^2} = c^2 \nabla^2 u(\underline{x},t) + F(\underline{x},t)$$

1D



Remarks:

* $\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$ divergence of the gradient

where does this come from physically speaking?

* Linear vs. Nonlinear wrt dependent variable(s)

All these examples are linear; significant for solution techniques

A nonlinear example

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u}$$

$\underline{u} = \underline{u}(\underline{x}, t)$ is fluid velocity

$p = p(\underline{x}, t)$ is fluid pressure

In 1D

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \quad \text{Burger's Eqn.}$$

$u = u(x, t)$

* Homogeneous vs. Non-homogeneous

$Q(\underline{x},t)=0$, $F(\underline{x},t)=0$ $\qquad$ $Q(\underline{x},t)\neq 0$, $F(\underline{x},t)\neq 0$

* Boundary conditions ; initial conditions ?

Start with the physical problem of heat conduction in a solid

Conduction in a solid $\Longrightarrow$ Heat Eqn.

Conduction <u>and</u> convection in a liquid/gas

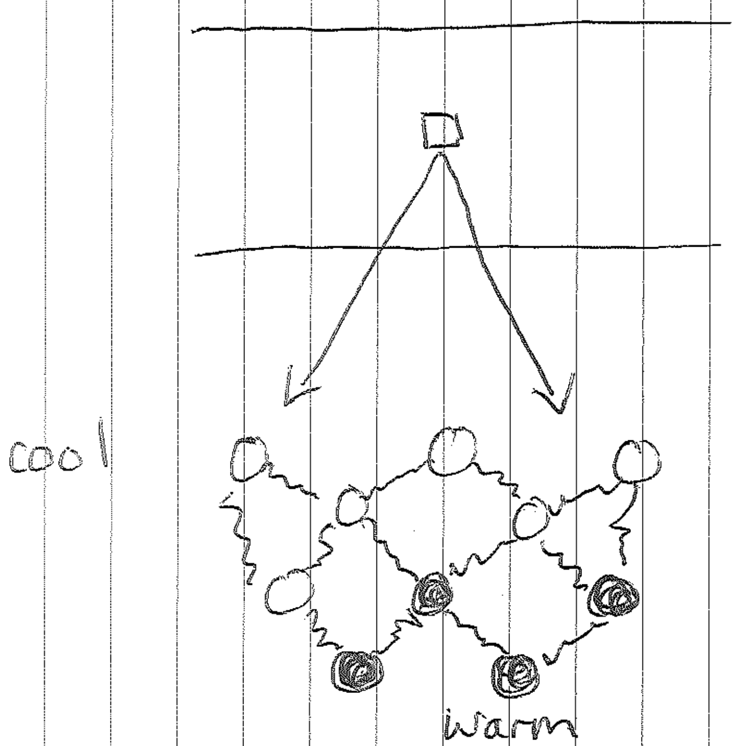
$\Longrightarrow$ Boussinesq Eqns.

Conduction in a solid

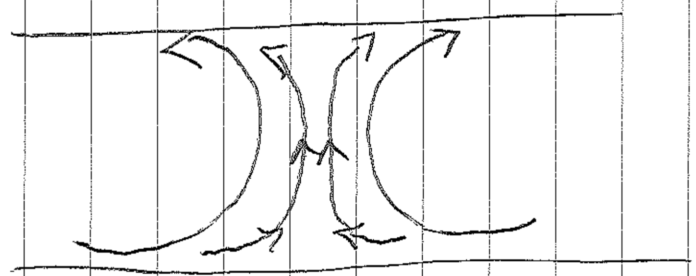
Convection in a fluid

$z = H$, T_c

$z = 0$, T_h



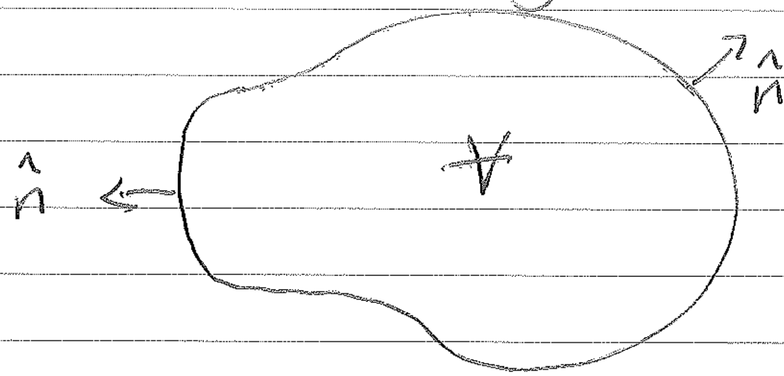
Diffusion is described by the Heat Eqn.; a linear equation



A fluid **Flows**

Convection in a fluid is described by the Boussinesq equations; nonlinear equations

Consider an arbitrary volume of solid in 3D



$\forall$ volume

$\hat{n}$ is the outward normal to the surface

Conservation of Internal Energy:

rate of change of energy in $\forall$ = energy in per unit time − energy leaving per unit time

+ energy generated inside $\forall$ by sources/sinks per unit time

All terms are $\frac{\Delta E}{\Delta t}$

Dimensional Analysis:

[ ] => "dimension of"

internal energy: [energy] = [Force · distance]

$$[\text{energy}] = \left[\frac{m\ell}{t^2}\cdot \ell\right] = \left[\frac{m\ell^2}{t^2}\right]$$

e.g. $\frac{kg\, m^2}{s^2}$ $\left\{\begin{array}{ccc} \text{same dimensions as} & & \\ \text{kinetic} & \text{and} & \text{potential} \\ \frac{1}{2}mv^2 & & mgh \end{array}\right\}$

It will be convenient to work with

e = internal energy density ; $e = e(\underline{x}, t)$

$$[e] = \left[\frac{\text{energy}}{\text{volume}}\right] = \left[\frac{m\ell^2}{t^2}\frac{1}{\ell^3}\right] = \left[\frac{m}{\ell t^2}\right]$$

"Control Volume" formulation in terms of energy density:

$$\frac{d}{dt}\int_{\forall} e\, d\forall = \frac{\text{energy in}}{\text{unit time}} - \frac{\text{energy out}}{\text{unit time}} + \frac{\text{energy generated}}{\text{unit time}}$$

$\uparrow$

$\iiint dx\,dy\,dz$ in Cartesian coordinates

$\boxed{\text{RHS terms}}$

$\dfrac{\text{energy in} - \text{energy out}}{\text{unit time}}$ is

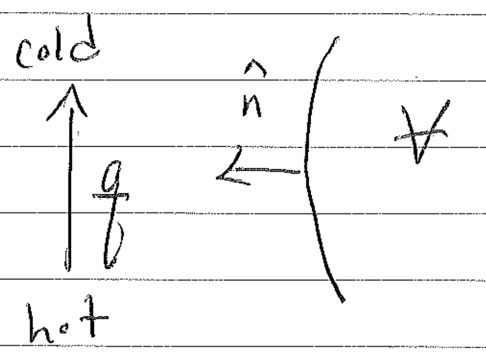
energy crossing the boundary of the volume, i.e. the surface enclosing the volume $\Rightarrow$

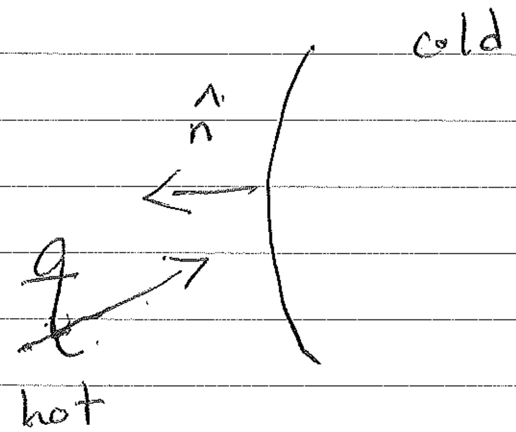
$$\oint_A q \cdot (-\hat{n})\, dA$$

dA is a surface element

where q is the heat flux vector

q is a vector with magnitude and direction because heat/energy flows from hot to cold





no energy enters $\forall$

$q \cdot (-\hat{n}) = 0$

energy enters $\forall$

$q \cdot (-\hat{n}) \neq 0$

The component of $\underset{\sim}{q}$ that matters is in the direction of $(-\hat{n})$. The sign is so that "energy in" is positive and "energy out" is negative.

$$\frac{\text{energy generated by sources/sinks}}{\text{unit time}}$$

$$\int_{\forall} Q(\underset{\sim}{x},t)\,d\forall \qquad \left(\begin{array}{c}\text{sources/sinks}\\ \text{inside } \forall\end{array}\right)$$

Finally we arrive at

$$\frac{d}{dt}\int_{\forall} e\,d\forall = \oint_A \underset{\sim}{q}\cdot(-\hat{n})\,dA$$

$$+ \int_{\forall} Q(\underset{\sim}{x},t)\,d\forall$$

$e(\underset{\sim}{x},t)$ is energy density
$\underset{\sim}{q}(\underset{\sim}{x},t)$ is the heat flux vector

$\boxed{\text{Next}}$

1. Dimensional Analysis: What are the dimensions of the heat flux vector q and the term representing sources/sinks Q?

2. Can we write this equation in terms of temperature $T(\underline{x}, t)$

   What is $e(\underline{x}, t)$, $q(\underline{x}, t)$ in terms of $T(\underline{x}, t)$?

3. Then how do we go from the "Control Volume" equation to the PDE?