Finish discussion of elementary numerical methods to solve 
\[
\frac{dy(t)}{dt} = f(t, y) \quad y(t_0) = y_0
\]

**FE:** \[\frac{\hat{y}_{n+1} - \hat{y}_n}{h} = f(t_n, \hat{y}_n) \quad \text{Explicit, one-step}\]

**BE:** \[\frac{\hat{y}_{n+1} - \hat{y}_n}{h} = f(t_{n+1}, \hat{y}_{n+1}) \quad \text{Implicit, one-step}\]

**ME = RK2**
\[
\hat{y}_{n+1} = \hat{y}_n + \frac{h}{2} \left\{ f(t_n, \hat{y}_n) + f(t_{n+1}, \hat{u}_{n+1}) \right\}
\]

\[
\hat{u}_{n+1} = \hat{y}_n + h f(t_n, \hat{y}_n)
\]

<table>
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<tr>
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<th>FE</th>
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<tr>
<td>local error</td>
<td>(O(h^2))</td>
<td>(O(h^2))</td>
<td>(O(h^3))</td>
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<tr>
<td>global error</td>
<td>(O(h))</td>
<td>(O(h))</td>
<td>(O(h^2))</td>
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An illustrative example (not a proof) using

\[ y' = y, \quad y(0) = 1 \]

Exact solution: \[ y = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \]

Let's examine RK2 with step size \( h \):

**Step #1**

\[
\begin{align*}
\tilde{y}_1 &= y_0 + \frac{h}{2} \left[ y_0 + \tilde{u}_1 \right], \quad \tilde{u}_1 = y_0 + hy_0 \\
\tilde{y}_1 &= y_0 + \frac{h}{2} \left[ y_0 + y_0 + hy_0 \right] \\
&= 1 + \frac{h}{2} \left[ 1 + 1 + h \right] = 1 + h + \frac{h^2}{2} = \tilde{y}_1(h)
\end{align*}
\]

Exact: \[ y(h) = e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots \]

The local error is \( O(h^3) \)
\[ \text{Step 2:} \quad \hat{y}_2 = \hat{y}_1 + \frac{h}{2} \left[ \hat{y}_1 + \hat{y}_2 \right], \quad \hat{u}_2 = \hat{y}_1 + h\hat{y}_2 \]

\[ \hat{y}_2 = \hat{y}_1 + \frac{h}{2} \left( \hat{y}_1 + \hat{y}_1 + h\hat{y}_1 \right) \]

\[ = 1 + h + \frac{h^2}{2} + \frac{h^3}{2} \left\{ a \left( 1 + h + \frac{h^2}{2} \right) + h \left( 1 + h + \frac{h^2}{2} \right) \right\} \]

\[ = 1 + 2h + 2h^2 + h^3 + \frac{h^4}{4} = \hat{y}_2(2h) \]

Exact: \[ y(2h) = 1 + 2h + \frac{2h^2}{2!} + \frac{(2h)^3}{3!} + \frac{(2h)^4}{4!} + \ldots \]

\[ = 1 + 2h + 2h^2 + \frac{8h^3}{6} + \ldots \]

The local error is \( O(h^3) \)

Now the same argument for the global error after \( n \) steps:

The biggest term: \( Cn h^3 \)

where \( n = \frac{(t_{\text{final}} - t_0)}{h} \) \( \Rightarrow \)

Global error \( \leq C' h^2 \left\{ \frac{C(t_{\text{final}} - t_0) h^3}{h} \right\} \)

or Global error \( O(h^2) \)
Now when we decrease the step size by a factor 2, we decrease the global error by a factor 4 (better than FE or BE!)

Example of how to use RK2:

\[ \frac{dy}{dx} = \frac{x^3}{y^2} \quad y(3) = 10 \]

\[ \hat{y}_{n+1} = y_n + \frac{h}{2} \left[ F(x_n, \hat{y}_n) + F(x_{n+1}, \hat{y}_{n+1}) \right] \]

\[ \hat{y}_{n+1} = y_n + h F(x_n, y_n) \quad F(x, y) = \frac{x^3}{y^2} \]

\[ \tilde{y}_{n+1} = \hat{y}_n + \frac{h}{2} \left\{ \frac{x_n^3}{y_n^2} + \frac{x_{n+1}^3}{\hat{y}_{n+1}^2} \right\} \]

\[ \hat{y}_{n+1} = \hat{y}_n + h \frac{x_n^3}{y_n^2} \quad \Rightarrow \]

\[ \tilde{y}_{n+1} = \hat{y}_n + \frac{h}{2} \left\{ \frac{x_n^3}{y_n^2} + \left( \frac{x_{n+1}^3}{(\hat{y}_n + h \frac{x_n^3}{y_n^2})^2} \right) \right\} \]
Summary: 1st order ODEs

\[ \frac{dy(x)}{dx} = f(x, y) \quad y(x_0) = y_0 \]

1. Existence & Uniqueness: Find an open interval containing \((x_0, y_0)\) where \(f\), \(\frac{df}{dy}\) are real and continuous.

2. Separable \( n(y) \frac{dy}{dx} = m(x) \)

3. Linear standard form

\[ y' + p(x)y = q(x) \]

Integrating factor \( M(x) = \exp \left\{ \int p(x) \, dx \right\} \)

4. Autonomous \( \frac{dy}{dx} = f(y) \) use graphical techniques (analyze critical points, slope fields).

5. Numerical methods \( \frac{dy}{dx} = f(x, y) \)

FE, BE, RK2
Now that we have thought about analytical, graphical, and numerical solution of a single 1st-order ODE, let's return to systems of coupled 1st-order ODEs.

Motivational Examples

Lecture 1: Temperature evolution in a system of coupled rooms

Population Dynamics: For more than one species, e.g., foxes and rabbits

\[
\frac{dx_1(t)}{dt} = F_1(t, x_1(t), x_2(t))
\]

\[
\frac{dx_2(t)}{dt} = F_2(t, x_1(t), x_2(t))
\]

\[x_1(t) = \text{population of rabbits}\]

\[x_2(t) = \text{number of foxes}\]

If the system is linear, and in the steady state with \( \frac{d}{dt} = 0 \), then we fall back on a system of linear algebraic equations.
Take time to investigate linear algebraic systems and return to strike evolution letter.

Consider dependent variables

\[ x_1(t), x_2(t), \ldots, x_n(t) \]

all functions of a single independent variable time \( t \), determined by \textit{linear} ODEs:

\[
\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1
\]

\[
\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2
\]

\vdots

\[
\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n - b_n
\]

\( n \) equations for \( n \) unknowns (typical of physical systems)

In general \( a_{ij} = a_{ij}(t) \) \( b_i = b_i(t) \)

\( i = 1, n \quad j = 1, n \)