Well-posedness for Laplace's equation with Dirichlet boundary conditions.

Idea: A problem is well posed if the solution varies a small amount for small changes in the boundary or initial data.

Consider \( \nabla^2 u = 0 \) inside \( R \)
\[ u = F(x) \] on the boundary of \( R \).

Now make a small change in boundary data
\( \Rightarrow \) there is a different solution \( v \) satisfying
\[ \nabla^2 v = 0 \text{ inside } R \]
\[ v = g(x) \text{ on the boundary of } R \]
with \( F(x) - g(x) \) small at all points on boundary.

Now consider \( w = u - v \) :
\( w \) satisfies \( \nabla^2 w = \nabla^2 (u - v) = \nabla^2 u - \nabla^2 v = 0 - 0 = 0 \)
where linearity has been used.

with \( w(x) = u(x) - v(x) = F(x) - g(x) = h(x) \)
specified on the boundary.

At all points in \( R \), the max/min principle gives
\[ h_{\text{min}} \leq W \leq h_{\text{max}} \]

\[ \min (F - g) \leq W \leq \max (F - g) \]

so we can make \( W \) in the interior as small as we want.

Explicit Example:

\[ \nabla^2 u(r, \theta) = 0, \quad 0 \leq r < a, \quad -\pi < \theta \leq \pi \]

\[ u(a, \theta) = f(\theta) \]

\[ f(\theta) = \cos 3\theta \quad q(\theta) = \cos 3\theta - \varepsilon \cos N\theta \]

\[ u(r, \theta) = \left( \frac{r}{a} \right)^3 \cos 3\theta \quad v(r, \theta) = \left( \frac{r}{a} \right)^3 \cos 3\theta \]

\[ = \varepsilon \left( \frac{r}{a} \right)^N \cos N\theta, \quad \varepsilon > 0 \]

\[ u(r, \theta) - v(r, \theta) = \varepsilon \left( \frac{r}{a} \right)^N \cos N\theta \]

\[ -\varepsilon \leq u(r, \theta) - v(r, \theta) \leq \varepsilon \]

Since \( \left( \frac{r}{a} \right)^N \leq 1 \), \( -1 \leq \cos N\theta \leq 1 \)
Uniqueness \[ \nabla^2 u = 0 \] Dirichlet b.c.s

Maximum Principle again

Suppose that there are 2 solutions \( u \) and \( v \) satisfying

\[ \nabla^2 u = 0 \text{ in } \Omega, \quad u = f(x) \text{ on the boundary of } \Omega \]

\[ \nabla^2 v = 0 \text{ in } \Omega, \quad v = f(x) \text{ on the boundary} \]

Max Principle tells us

\[ 0 \leq w = u - v \leq 0 \implies u = v \]

at all interior points

Haplace's Equation with Dirichlet boundary condition

* well-posed
* unique soln.
Solvability condition for flux boundary condition:
\[-K \nabla u(x, t) \cdot \mathbf{n} \text{ specified on boundary}\]

\[\nabla^2 u = 0 \text{ in interior}\]

Integrate over the domain (2D)

\[
\int \nabla^2 u \, dx \, dy = \oint \nabla \cdot (\nabla u) \, dx \, dy = 0
\]

\[= \oint \nabla u \cdot \mathbf{n} \, ds\]

\[= 0\]

\[\Rightarrow \text{net heat flux through boundary must be zero}\]

\[\text{otherwise no solution}\]
Laplace's Eqn. outside the circular cylinder and what's this have to do with fluid flow.

We said our heat eqn. is for heat conduction in a solid... we now mention fluid flow around an airfoil.

Let's say we wanted to consider Conservation of Energy in an arbitrary volume of fluid.

Now there is a fluid velocity that can carry heat across the boundary.

At the macroscopic level, there is a flux across the boundary.

\[ \int_A u(x,t) e(x,t) \cdot (-\hat{n}) \, dA \]
Check units: \( \left[ \frac{L}{t^3} \right] \left[ \frac{mL^2}{t^3} \right] \left[ L^2 \right] = \left[ \frac{mL^2}{t^3} \right] \)

For fluid flow we need conservation of mass, momentum, energy and a thermodynamic relation.

\( \Rightarrow 6 \) equations total for

6 scalar quantities: \( T(x,t) \), \( p(x,t) \), \( \rho(x,t) \)
and 3 components of velocity \( u(x,t) \)

Consider conservation of mass \( \varepsilon 20.5.17 \)

\[ \frac{d}{dt} \int_A \rho \, dA = \int_A u \rho (\hat{n}) \, dA \]
\[ = -\int_A \nabla \cdot (\rho \mathbf{u}) \, dA \]

Now shrink to a point \( \Rightarrow \)

\[ \frac{\partial}{\partial t} \rho(x,t) + \nabla \cdot (\rho(x,t) \mathbf{u}(x,t)) = 0 \]
If $Q$ is constant $\Rightarrow$

\[ \nabla \cdot \mathbf{u} = 0 \]

Next, repeat for momentum, energy and again assuming constant density $\Rightarrow$

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_c} \nabla p + \mathbf{g} \]

\[ \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \cdots \]

Complicated coupled, nonlinear PDEs.

Let's make the very strange and very special assumption that the flow satisfies a condition

\[ \nabla \times \mathbf{u} = 0 \]

The flow is "irrotational"

What could this mean?