Conservation of mass, momentum, energy for a constant density flow:

mass: \( \nabla \cdot \mathbf{u} = 0 \)

momentum: \( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \mathbf{g} \) (per unit mass, per volume)

energy: \( \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \cdots \)

For flow over a 2D airfoil, we replace this system (coupled nonlinear PDEs) by

\( \nabla \cdot \mathbf{u} = 0 \)

\( \nabla \times \mathbf{u} = 0 \)

\( p + \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} = p_0 \) constant \( \left[ \frac{\text{force}}{\text{area}} \right] \)

\( \nabla \times \mathbf{u} = 0 \) : the flow is "irrotational"

What does this mean?
Think about a 2D parcel of fluid

\[ u = u(x, y) = u(x, y) \hat{x} + v(x, y) \hat{y} \]

In Cartesian:

\[ \nabla \times u = \mathbf{e}_z \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

If \( v(x) \) increases with \( x \); \( u(y) \) increases with \( y \)

\[ \nabla \times u \] is associated with local spin

Examples of irrotational flows:

1. \( u = u_0 \hat{x} \) \quad \rightarrow \quad \text{uniform flow}

2. \[ u = u_r \hat{r} + u_\theta \hat{\theta} = (u_r, u_\theta) \]

\[ u_\theta = \frac{k}{r}, \quad u_r = 0, \quad r > 0 \]

parcels not rotating locally as they rotate
\[ u = u_r \hat{r} + u_\theta \hat{\theta} + u_\phi \hat{\phi} \]
\[
\nabla \times \mathbf{u} = \frac{1}{r} \left( \frac{\partial u_\phi}{\partial \theta} - \frac{\partial u_\theta}{\partial \phi} \right) \hat{r} + \left( \frac{\partial u_r}{\partial \phi} - \frac{\partial u_\phi}{\partial r} \right) \hat{\theta} + \left( \frac{2}{r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right) \hat{\phi}
\]
\[
= 0 \hat{r} + 0 \hat{\theta} + 0 \hat{\phi}
\]

What is NOT irrotational?

1. Simple shear \( \mathbf{u} = u_x \hat{x} + v_y \hat{y} = u_0 y \hat{x} \)

\[
\nabla \times \mathbf{u} = -\frac{\partial u_y}{\partial x} = -u_0
\]

Since a fluid at the boundary will have the same velocity as the boundary \( \Rightarrow \)

\[ \text{boundaries (viscosity) create vorticity!} \]
Ignore viscosity for flow over an airfoil. Fluid flow is called to "slip" on the boundary. Irrotational flow stays irrotational $\Rightarrow$ slip on boundary.

$2D$ irrotational flow in Cartesian:

* mass $\nabla \cdot \mathbf{u} = 0 \Rightarrow$

$$
\mathbf{u} = \frac{dy}{dx} \hat{x} - \frac{dx}{dy} \hat{y} = (u, v)
$$

$$
\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial x} \left( \rho \frac{dy}{dx} \right) - \frac{1}{\rho} \frac{\partial}{\partial y} \left( \rho \frac{dx}{dy} \right)
$$

$$= 0$$

* Then $\nabla \times \mathbf{u} = 0$

$$
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 = \frac{1}{\rho} \left( \frac{\partial}{\partial x} \left( -\frac{dy}{dx} \right) \right) - \frac{1}{\rho} \frac{\partial}{\partial y} \left( \frac{dx}{dy} \right)
$$

$$\nabla^2 \psi = 0$$
In Polar coordinates \( u = u_r \hat{r} + u_\theta \hat{\theta} \)

\[
u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} \Rightarrow \nabla^2 \psi(r, \theta) = 0
\]

Check:

\[
\nabla \cdot u = \frac{1}{r} \frac{\partial}{\partial r} \left( r u_r \right) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \\
= \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{r} \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( -\frac{\partial \psi}{\partial r} \right) \\
= \frac{1}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial r} = 0
\]

A streamline \( \psi(r, \theta) = \text{constant} \) is everywhere tangent to the velocity vector.

We've already solved \( \nabla^2 \psi = 0 \) with \( \psi(a, \theta) = F(\theta) \) but now we are outside the circle.

\[
\psi(r, \theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left( a_n \cos n\theta + b_n \sin n\theta \right) \left( C_1 r^n + C_2 r^{-n} \right)
\]
What should we take for $F(\theta)$? What other conditions do we need?

Uniform flow in $\hat{x}$

\[ \rightarrow \]

\[ \rightarrow \]

no flow through the boundary: $u_r(a, \theta) = 0$

1. Uniform Flow in $\hat{x}$: $\hat{y} = u_0 y = u_0 \sin \theta$

\[ \Rightarrow \lim_{r \to \infty} \hat{y}(r, \theta) = u_0 \sin \theta \]

suggests keep only $n = 1$, and the $\sin \theta$ term for $r \to \infty$

\[ \hat{y}(r, \theta) = A_0 + B_0 \ln r + u_0 \sin \theta + A r^{-1} \cos \theta + B r^{-1} \sin \theta \]

check:\n
\[ \lim_{r \to \infty} \hat{y}(r, \theta) \sim u_0 \sin \theta \]

because the constant term and $\ln r$ terms are subdominant as $r \to \infty$

\[ \text{[In}(r) \text{ increases less rapidly than } r \text{ as } r \to \infty] \]

[ok]
(2) Now apply no flow through boundary

\[ u_r \big|_{r=a} = \frac{1}{r} \frac{\partial u}{\partial \theta} \big|_{r=a} = 0 \quad \left[ \frac{\partial u}{\partial \theta} \big|_{r=a} = 0 \right] \]

\[ \frac{\partial u}{\partial \theta} \big|_{r=a} = \left( u_0 r + B_1 r^{-1} \right) \cos \theta \big|_{r=a} - A_1 r^{-1} \sin \theta \big|_{r=a} = 0 \]

\[ (u_0 a + B_1/a) \cos \theta - A_1/a \sin \theta = 0 \]

\[ \Rightarrow u_0 a + B_1/a = 0 \quad A_1 = 0 \quad \text{sine} \]

\[ \sin \theta, \cos \theta \text{ are linearly independent} \]

\[ \Rightarrow \Psi(r, \theta) = A_0 + B_0 \ln r + u_0 (r - a^2/r) \sin \theta \]

(3) Now what \( F(\theta) \) do we take?

Recall \( \Psi(a, \theta) = F(\theta) \)

\[ \Psi(a, \theta) = A_0 + B_0 \ln a + 0 = \text{constant}! \]

So the condition at \( r = \infty \) and the no flow through body force \( F(\theta) \) constant, but what constant?
Notice that there is an arbitrary constant in \( \Psi \) that is not physical because \( \Psi \) is an integral of the (physical) velocity field.

\[
\{ \text{Recall } u_r = \frac{1}{r} \frac{\partial \Phi}{\partial \theta}, \ u_\theta = -\frac{\partial \Phi}{\partial r} \}
\]

Integrate to find \( \Phi \).

\[ \rightarrow \text{We can specify whatever we want for the constant value of } \Psi(a, \theta) \]

\[ \rightarrow \text{Choose zero for simplicity} \]

\[ \rightarrow \Psi(a, \theta) = f(\theta) = \text{constant} = 0 \]

is a perfectly "good" choice. \( \rightarrow A_0 = -B_0 \eta \)

Notice

\[ \lim_{r \to \infty} \Psi(r, \theta) \sim u_\theta \text{sine} \]

\[ \Psi(a, \theta) = 0 \text{ is not enough for a unique solution because of the subdominant growth terms } \]

\[ [\ln(r)] \]
Finally \( \psi(r, \theta) = B_0 \ln(\frac{a}{r}) + u_0 \left( r - \frac{a^2}{r} \right) \sin \theta \), with 1 unknown coefficient \( B_0 \).

Luckily we have one more physical consideration: the circulation around the airfoil is related to the “slip” velocity (tangential velocity) around the airfoil.

\[
\text{Circulation at } r = a: \quad \oint_{r=a} u_0 \, \text{d} \theta
\]

Without viscosity, we have allowed any slip \( u_0(a, \theta) \) and therefore any circulation.
\[ u_0 = -\frac{2\gamma}{r} = -\sum B_0 \frac{1}{a} + u_0 \sin \theta + u_0 \frac{a^2}{r} \sin \theta \]

\[ u_0 \bigg|_{r=a} = -\frac{B_0}{a} + 2u_0 \sin \theta \quad \text{not determined because } B_0 \text{ unknown} \]

Circulation at \( r = a \):
\[ \oint_{\theta} u_0 \bigg|_{r=a} \ a \ d\theta \]
\[ = \oint_{\theta} -\frac{B_0}{a} + 2u_0 \sin \theta \ a \ d\theta = -2\pi B_0 = \Gamma \]

\( \Gamma \) same for all closed contours in flow.

What value of circulation should we choose?

Now need to invoke viscosity!

\[
\begin{array}{ccc}
\text{No} & \text{Yes} & \text{No} \\
\text{viscosity doesn't like sharp gradients of velocity} & \text{YES is a specific, negative value of circulation} &
\end{array}
\]
Summary of the flow over a circular cylinder

This is a special case of solving \( \nabla^2 u = 0 \) with \( u(a, \theta) = f(\theta) \); we want \( u(r, \theta) \) outside.

Here we call it \( \nabla^2 \Phi = 0 \)

The conditions:
1. Uniform flow at \( r \to \infty \)
2. No flow through the body

Gives \( f(\theta) = \text{constant} \) which we choose to be zero because \( \Phi \) is an integral of physical velocity \( \Rightarrow \) the actual value of the constant has no physical meaning.

We allowed any slip velocity (the picture above is only one special case), the physical case with stagnation points at
places that lead to zero circulation \( M_a \)

\[
M_a = -2\pi B_0 = \int_0^{2\pi} u_0 \bigg|_{r=a} \, d\theta
\]

But the value of \( M_a \) we expect (physically) is negative

\( \mu > 0 \) \quad \mu < 0 \quad \mu < 0

\[
\downarrow
\]

viscosity chooses this circulation corresponding to stagnation point at the sharp trailing edge of Kutta-Joukowski condition

negative circulation
\[ \psi(r, \theta) = B_0 \ln \left( \frac{r}{a} \right) + u_0 \left( r - \frac{a^2}{r} \right) \sin \theta \]

\[ u_0 = -\frac{\mu_0}{\varepsilon} \quad ; \quad u_0 \bigg|_{r=a} = -\frac{B_0}{a} + \alpha u_0 \sin \theta \]

\[ \Gamma_a = \int_0^{2\pi} u_0 \bigg|_{r=a} \ a \ d\theta = -2\pi B_0 \quad ( \text{Kolka-Jockers} ) \]
L = F \cdot \hat{y} \quad D = F \cdot \hat{x}

Normally both viscous and pressure forces, but here only pressure forces.

Energy Eqn: \( \rho + \frac{1}{2} u \cdot u = \text{constant} = \rho_0 \)

Pressure is \( \int_{\text{area}} \frac{\text{force}}{\text{area}} \) \( \Rightarrow \) integrate over area to get force.

at \( r = a \) fixed

\[
L = \left( \int_{c} b \cdot p_a \cdot (-\hat{n}) \, ds \right) \cdot \hat{y}
\]

\[
D = \left( \int_{c} b \cdot p_a \cdot (-\hat{n}) \, ds \right) \cdot \hat{x}
\]

\[
p_a = (\rho_0 - \frac{1}{2} U_0^2) \quad r = a \quad \text{no flow through cylinder}
\]

\[
U_0 = -\frac{dy}{dr}
\]
\[ \hat{n} \cdot \hat{y} = -\sin \theta \]

\[ \hat{n} \cdot \hat{x} = -\cos \theta \]

\[ l = b \int_{0}^{\pi} \rho a (-\sin \theta) \, d\theta = -(\rho a) b \]

where \( l = -2\pi B_0 \) negative

\[ D = b \int_{0}^{\pi} \rho a (-\cos \theta) \, d\theta = 0 \]

"D'Alembert's paradox"

Do these calculations yourself! Homework 06.