$\boxed{\text{Math 322 Lecture 11}}$

Conservation of mass, momentum, energy for a constant density flow:

mass: $\underline{\nabla} \cdot \underline{u} = 0$

momentum: $\frac{\partial u}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho_0} \nabla p + g$

(per unit mass, per volume)

energy: $\frac{\partial T}{\partial t} + (\underline{u} \cdot \underline{\nabla})T = \ldots$

For flow over a 2D airfoil, we replace this system (coupled nonlinear PDEs) by

$$\underline{\nabla} \cdot \underline{u} = 0$$

$$\underline{\nabla} \times \underline{u} = 0$$

$$p + \tfrac{1}{2}\rho_0 \underline{u} \cdot \underline{u} = p_0 \quad \text{constant} \quad \left[\frac{\text{force}}{\text{area}}\right]$$

$\underline{\nabla} \times \underline{u} = 0$: the flow is "irrotational"

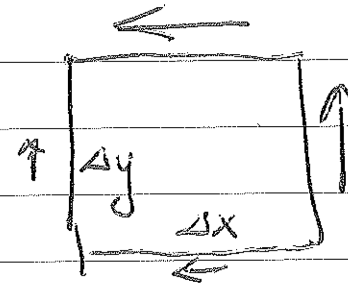
What does this mean?

Think about a 2D parcel of fluid

$$\underline{u} = \underline{u}(x,y) = u(x,y)\,\hat{x} + v(x,y)\,\hat{y}$$

In Cartesian: $\underline{\nabla} \times \underline{u} = \hat{z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

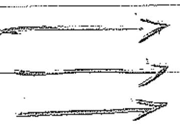
If $v(x)$ increases with x ; $u(y)$ increases with y



$\underline{\nabla} \times \underline{u}$ is associated with local spin

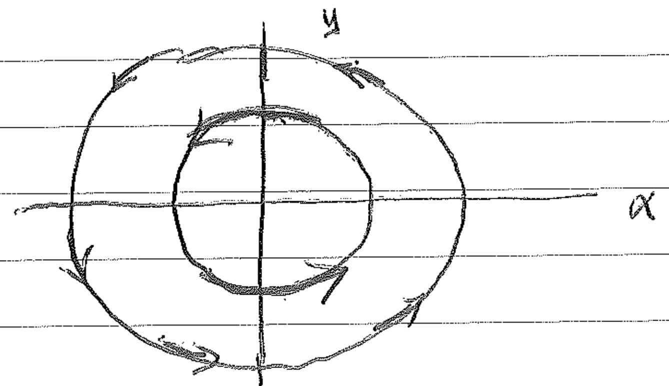
Examples of irrotational flows:

① $\underline{u} = U_0\,\hat{x}$ uniform flow



② $\underline{u} = u_r\,\hat{r} + u_\theta\,\hat{\theta} = (u_r, u_\theta)$

$u_\theta = \frac{k}{r}$, $u_r = 0$ $r > 0$



parcels not rotating locally as they rotate

$\underline{u} = u_r \hat{r} + u_\theta \hat{\theta} + u_z \hat{z}$:

$$\underline{\nabla} \times \underline{u} = \frac{1}{r}\left(\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\theta}$$

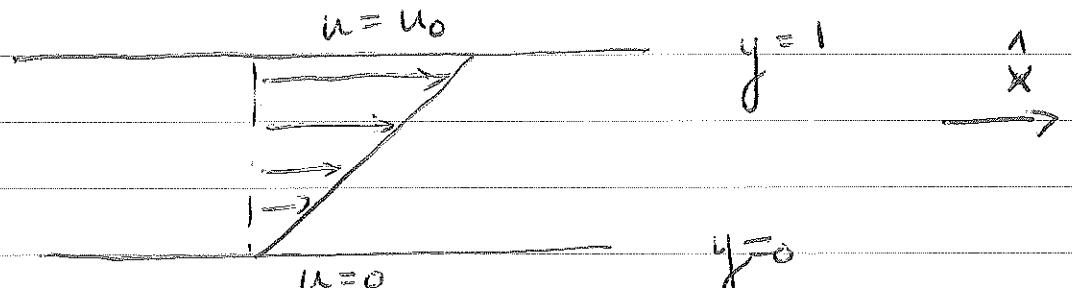
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{z}$$

$$= 0\hat{r} + 0\hat{\theta} + 0\hat{z}$$

What is NOT irrotational?

① Simple shear $\quad \underline{u} = u\hat{x} + v\hat{y} = u_0 y \hat{x}$

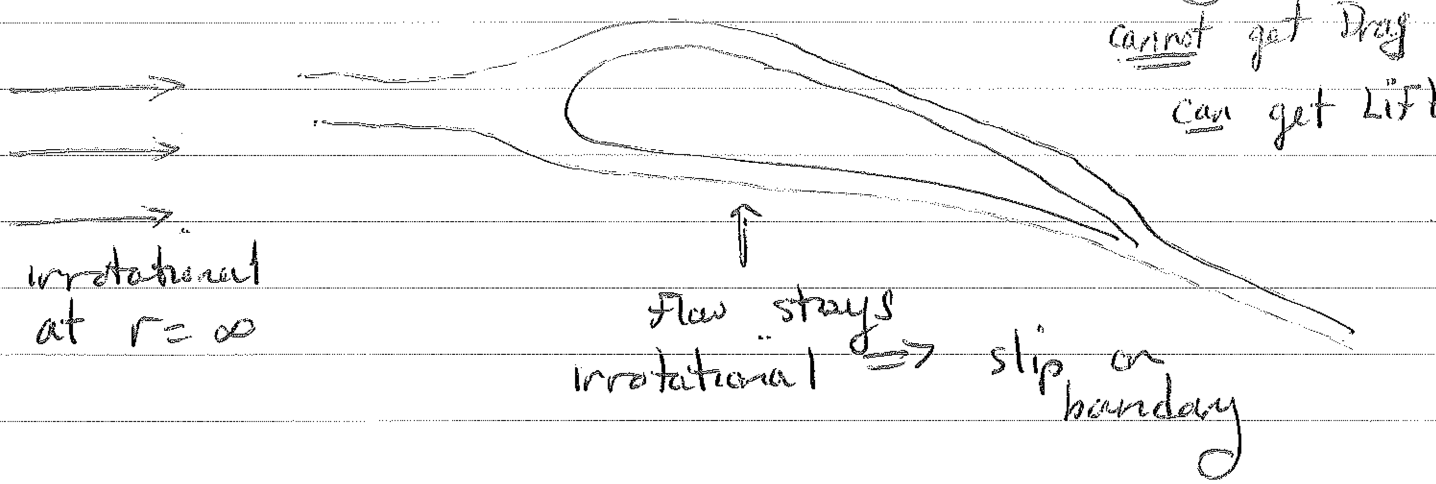
$$\underline{\nabla} \times \underline{u} = -\frac{\partial u}{\partial y} = -u_0$$



Since a fluid at the boundary will have the same velocity as the boundary $\Rightarrow$

boundaries (viscosity) create vorticity!

Ignore viscosity for flow over an airfoil ; all fluid to "slip" on the boundary



2D irrotational flow in Cartesian

* mass $\nabla \cdot \underline{u} = 0 \Rightarrow$

$$\underline{u} = \frac{\partial \psi}{\partial y}\hat{x} - \frac{\partial \psi}{\partial x}\hat{y} = (u, v)$$

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

$$= 0$$

* Then $\nabla \times \underline{u} = 0$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$$

$$= -\nabla^2 \psi = 0$$

in Polar Coordinates $\underline{u} = u_r \hat{r} + u_\theta \hat{\theta}$

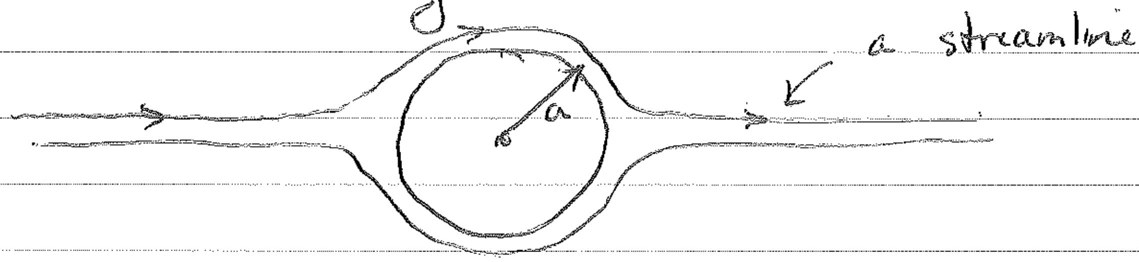
$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} \implies \nabla^2 \psi(r,\theta) = 0$$

Check: $\underline{\nabla} \cdot \underline{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{r}\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{-\partial \psi}{\partial r}\right)$$

$$= \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} + -\frac{1}{r}\frac{\partial^2 \psi}{\partial \theta \partial r} = 0 \quad \ldots$$

A streamline $\psi(r,\theta)$ = constant is everywhere tangent to the velocity vector

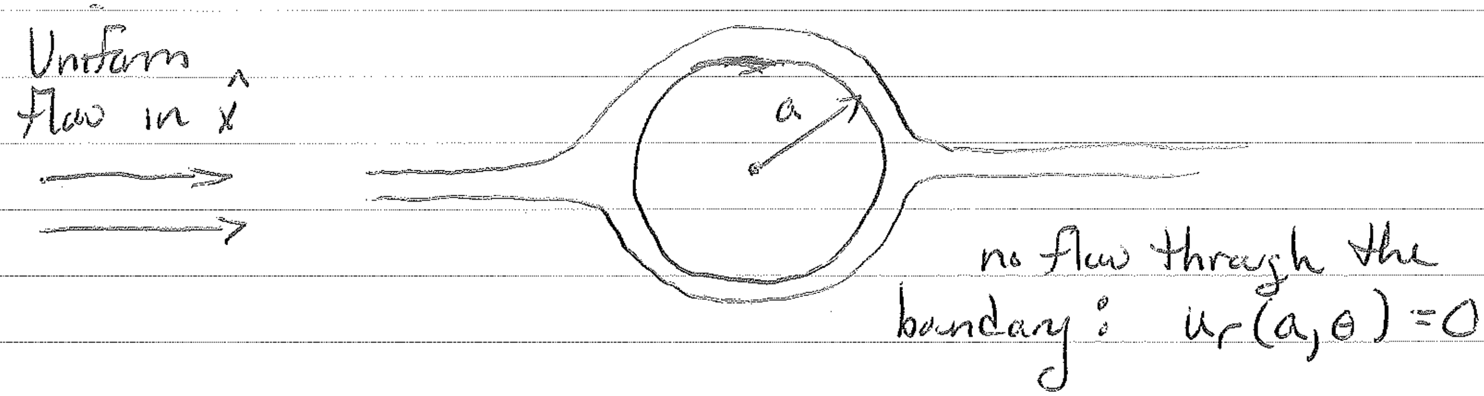


We've already solved $\nabla^2 \psi = 0$ with $\psi(a,\theta) = F(\theta)$ but now we are outside the circle

$$\psi(r,\theta) = A_0 + B_0 \ln r$$

$$+ \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_1 r^n + c_2 r^{-n})$$

What should we take for $F(\theta)$?
What other conditions do we need



① Uniform flow in $\hat{x}$: $\psi = U_0 y = U_0 r \sin\theta$

$$\Rightarrow \lim_{r \to \infty} \psi(r,\theta) = U_0 r \sin\theta$$

suggests keep only $n=1$, and the $\sin\theta$ term for $r \to \infty$

$$\psi(r,\theta) = A_0 + B_0 \ln r + U_0 r \sin\theta + A_1 r^{-1} \cos\theta + B_1 r^{-1} \sin\theta$$

check: $\lim_{r \to \infty} \psi(r,\theta) \sim U_0 r \sin\theta$

because the constant term and $\ln r$ terms are subdominant as $r \to \infty$

[$\ln(r)$ increases less rapidly than r as $r \to \infty$]

ok

(2) Now apply no flow through boundary

$$u_r\big|_{r=a} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\bigg|_{r=a} = 0 \qquad \left[\frac{\partial\psi}{\partial\theta}\bigg|_{r=a} = 0\right]$$

$$\frac{\partial\psi}{\partial\theta}\bigg|_{r=a} = \left(u_0 r + B_1 r^{-1}\right)\cos\theta\big|_{r=a} - A_1 r^{-1}\sin\theta\big|_{r=a} = 0$$

$$\left(u_0 a + B_1/a\right)\cos\theta - A_1/a \sin\theta = 0$$

$\Rightarrow u_0 a + B_1/a = 0 \qquad A_1 = 0$ since

$\sin\theta$, $\cos\theta$ are linearly independent

$$\Rightarrow \psi(r,\theta) = A_0 + B_0 \ln r + u_0\left(r - a^2/r\right)\sin\theta$$

(3) Now what $F(\theta)$ do we take?

Recall $\psi(a,\theta) = F(\theta)$

$\psi(a,\theta) = A_0 + B_0 \ln a + 0 = \text{constant}$!

So the condition at $r=\infty$ and the no flow through body

force $F(\theta)$ constant, but

what constant?

Notice that there is an arbitrary constant in ψ that is not physical because ψ is an integral of the (physical) velocity field

$$\left\{ \text{Recall} \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} \right\}$$

integrate to find ψ

$\Rightarrow$ We can specify whatever we want for the constant value of $\psi(a,\theta)$

$\Rightarrow$ choose zero for simplicity

$\Rightarrow$ $\psi(a,\theta) = f(\theta) = \text{constant} = 0$ is a perfectly "good" choice. $\Rightarrow A_0 = -B_0 \ln a$

Notice

$$\lim_{r\to\infty} \psi(r,\theta) \sim U_0 r \sin\theta.$$

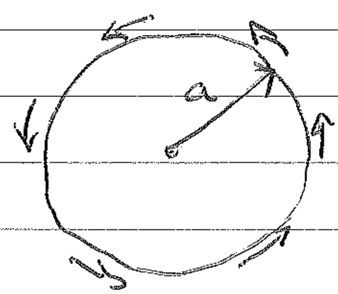
$\psi(a,\theta) = 0$ is not enough for a unique solution because of the subdominant growth terms

$[\ln(r)]$

Finally $\psi(r,\theta) = B_0 \ln(r/a) + U_0\left(r - \frac{a^2}{r}\right)\sin\theta$

with 1 unknown coefficient B_0.

Luckily we have one more physical consideration: the circulation around the airfoil is related to the "slip" velocity (tangential velocity) around the airfoil.



Circulation at $r=a$:

$$\int_0^{2\pi} u_\theta\Big|_{r=a} a\, d\theta$$

Without viscosity, we have allowed any slip $u_\theta(a,\theta)$ and therefore any circulation

$$u_\theta = -\frac{\partial \psi}{\partial r} = -\left\{ B_0 \frac{1}{a} \frac{1}{r/a} + U_0 \sin\theta + U_0 \frac{a^2}{r^2} \sin\theta \right\}$$

$$u_\theta \Big|_{r=a} = -\frac{B_0}{a} + 2U_0 \sin\theta$$ not determined because B_0 unknown

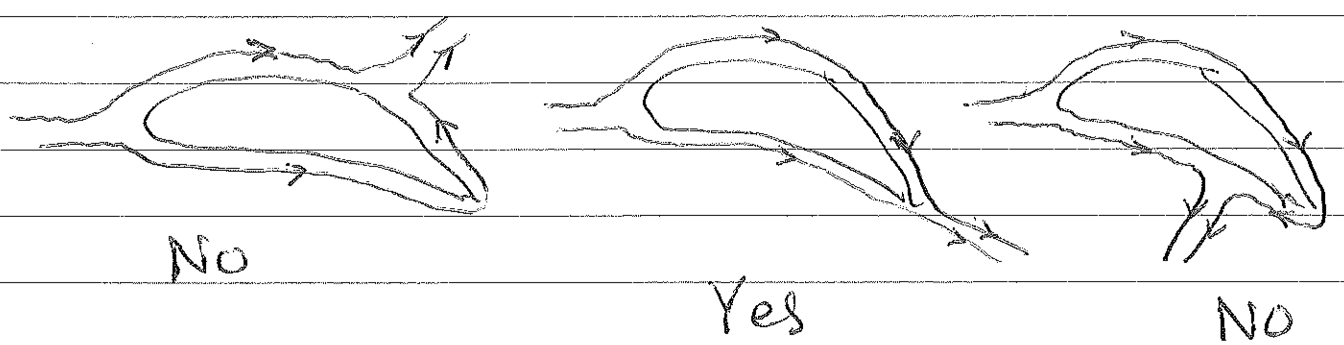
Circulation at $r=a$: $\int_0^{2\pi} u_\theta \big|_{r=a} \, a \, d\theta$

$$= \int_0^{2\pi} \left\{ -\frac{B_0}{a} + 2U_0 \sin\theta \right\} a \, d\theta = -2\pi B_0 = \Gamma$$

{same for all closed contours in flow}

What value of circulation Γ should we choose?

Now need to invoke viscosity!



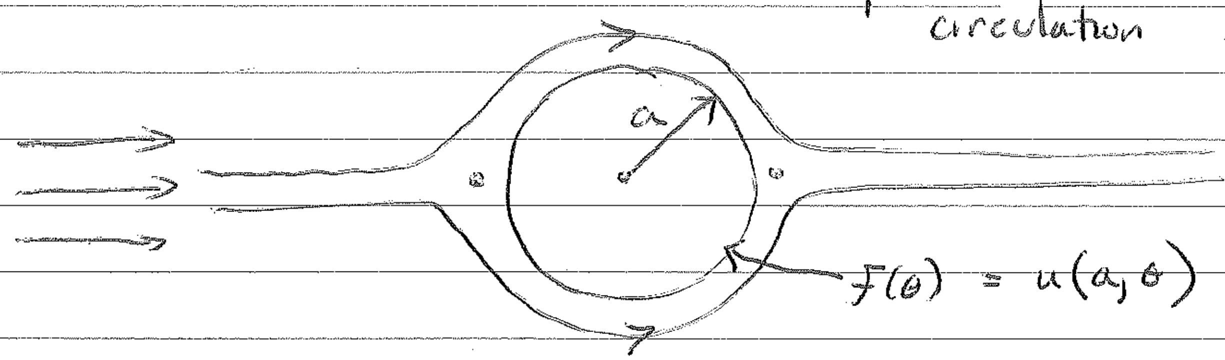
viscosity doesn't like sharp gradients of velocity!

YES is a specific, negative value of circulation

Math 322 Lecture 12

Summary of the flow over a circular cylinder

picture is for zero circulation Γ_a



This is a special case of solving $\nabla^2 u = 0$ with $u(a,\theta) = f(\theta)$; we want $u(r,\theta)$ outside

Here we call it $\nabla^2 \psi = 0$

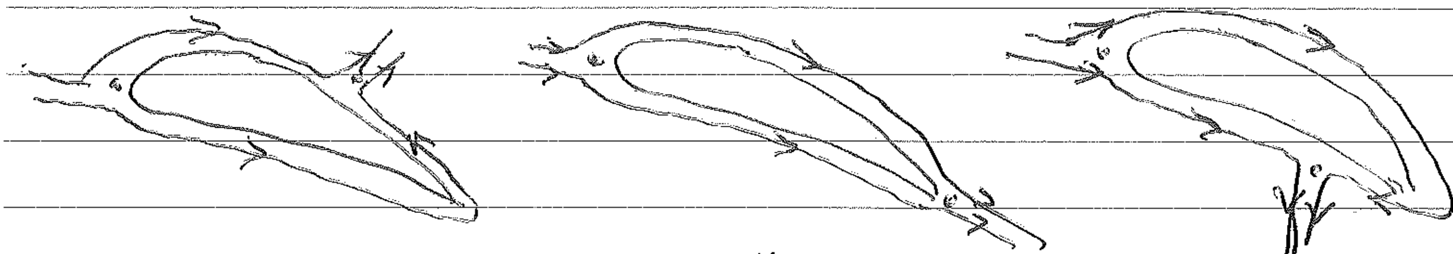
The conditions ① Uniform flow at $r \to \infty$ ② no flow through the body

gives $f(\theta)$ = constant which we choose to be zero because ψ is an integral of physical velocity $\Rightarrow$ the actual value of the constant has no physical meaning

We allowed any slip velocity (the picture above is only one special case; the physical case with stagnation points at places that lead to zero circulation Γ_a)

$$\Gamma_a = -2\pi B_0 = \int_0^{2\pi} u_\theta\Big|_{r=a} a\, d\theta$$

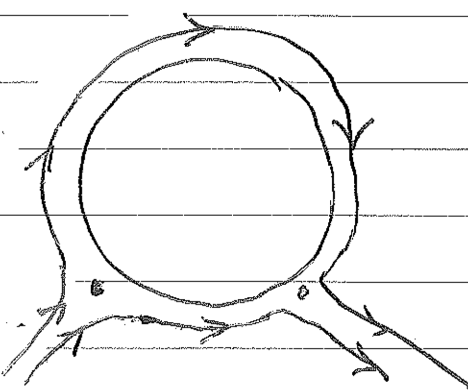
But the value of Γ_a we expect (physically) is negative



$\Gamma > 0$ $\qquad$ $\Gamma < 0$ $\qquad$ $\Gamma < 0$

↓

viscosity chooses this circulation corresponding to stagnation point at the sharp trailing edge ; Kutta-Joukowsy Condition



negative circulation

③

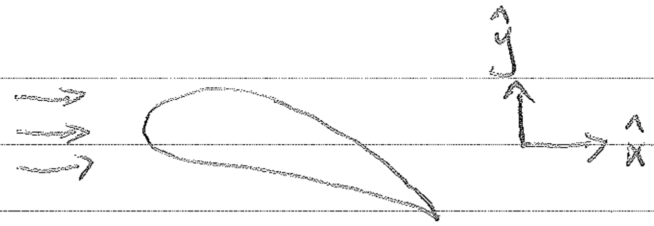
$$\psi(r,\theta) = B_0 \ln(r/a) + U_0\left(r - a^2/r\right)\sin\theta$$

$$u_\theta = -\frac{\partial\psi}{\partial r} \;;\; u_\theta\Big|_{r=a} = -\frac{B_0}{a} + 2U_0\sin\theta$$

$$\Gamma_a = \int_0^{2\pi} u_\theta\Big|_{r=a} a\,d\theta = -2\pi B_0 \quad \left(\begin{matrix} <0 \quad \text{by} \\ \text{Kutta-Joukowsky} \end{matrix}\right)$$

Lift and Drag

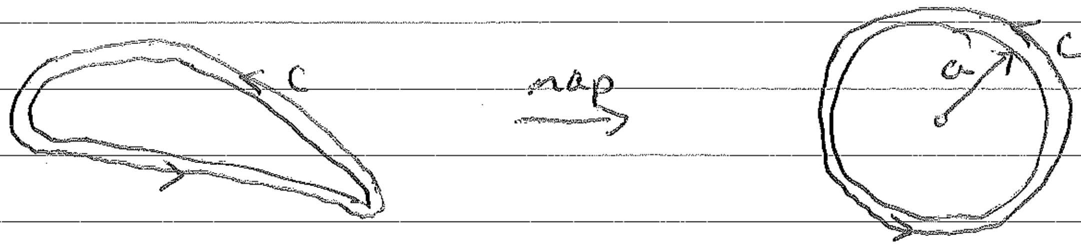
$L = \underline{F} \cdot \hat{y} \qquad D = \underline{F} \cdot \hat{x}$



Normally both viscous and pressure forces, but here only pressure forces

Energy Eqn: $p + \frac{1}{2} \underline{u} \cdot \underline{u} = \text{constant} = p_0$

pressure is $\left[\text{force}/\text{area} \right] \Rightarrow$ integrate over area to get force



at $r = a$ fixed

$$L = \left(b \oint_c p_a (-\hat{n}) \, dS \right) \cdot \hat{y}$$

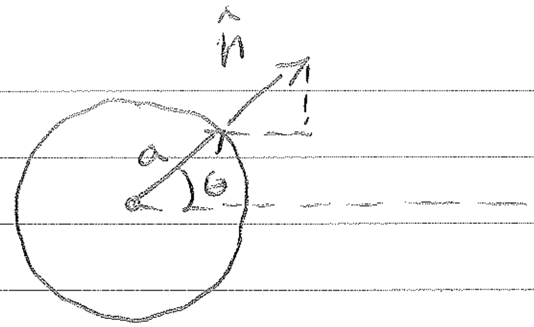
b = width out of board

$$D = \left(b \oint_c p_a (-\hat{n}) \, dS \right) \cdot \hat{x}$$

{2D airfoil}

$p_a = \left(p_0 - \frac{1}{2} u_\theta u_\theta \right)_{r=a}$ no flow through cylinder

$$u_\theta = -\frac{\partial \psi}{\partial r}$$



$$-\hat{n}\cdot\hat{y} = -\sin\theta$$

$$-\hat{n}\cdot\hat{x} = -\cos\theta$$

$$L = b\int_0^{2\pi} p a(-\sin\theta)\, a\, d\theta = -\rho U_0 \Gamma b$$

where $\Gamma = -2\pi B_0$ negative

$$D = b\int_0^{2\pi} p a\,(-\cos\theta)\, a\, d\theta = 0$$

"D'Alembert's paradox"

Do these calculations yourself! Homework ...