Goal: To understand when
\[ F(x) \approx a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \] "makes sense."

1. Does the infinite series converge?
2. Does it converge to \( f(x) \)?

Definition: A function is piecewise smooth on \([-L, L]\) if:
* \([-L, L]\) can be divided into pieces
* In each piece, \( f(x) \) and \( \frac{df(x)}{dx} \) are continuous.

Function values and derivatives must be finite.
There is a jump discontinuity in \( f(x) \) at 
\[ x_0 = -\frac{1}{2} \quad \text{and} \quad x_0 = \frac{3}{4} \]

There is a jump discontinuity in \( \frac{d^2f(x)}{dx} \) at \( x_0 = 0 \)

There are 4 intervals: 
\[ (-1, -\frac{1}{2}) \quad (-\frac{1}{2}, 0) \quad (0, \frac{3}{4}) \quad (\frac{3}{4}, 2) \]

Inside each open interval, \( f(x) \) and \( \frac{d^2f(x)}{dx} \) are continuous \( \Rightarrow \) the function is piecewise smooth in \(-1 \leq x \leq 2\)

\( f(x) = x^{1/3} \) is not piecewise smooth on \(-1 \leq x \leq 2\) because its derivative is not finite at \( x_0 = 0 \).

\[ f(x) \quad \text{Finite limits of } \frac{d^2f(x)}{dx} \quad \text{are required!} \]

Not allowed.
In Chapter 3 we always assume that \( f(x) \) is piecewise smooth in \( [-1,1] \).

If not stated explicitly, it is implicit...
What is a periodic extension of \( f(x) \)?
outside \( -1 \leq x \leq 1 \)

Repeat the shape of \( f(x) \), with period \( 2L \)

\[
\text{Convergence Theorem: If } f(x) \text{ is piecewise smooth on } -L \leq x \leq L, \text{ then the series corresponding to } \]
\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \]
converges to

1. the periodic extension of \( f(x) \), wherever the periodic extension is continuous
2. \( \frac{1}{2} \left[ f(x^+) + f(x^-) \right] \) wherever the periodic extension has a jump discontinuity
In the example plot above

Jump discontinuities at \( x = nL \)

at \( x_0 = 0 \) :  \( \frac{f(0^-) + f(0^+)}{2} = 2 \)

etc.

at \( x_0 = L \) :  \( \frac{f(L^-) + f(L^+)}{2} = \frac{1}{2} \)

Some straightforward facts:

If \( f(x) \) is even, only the \( a_n \)'s will be nonzero (therefore we can start with a cosine series).

If \( f(x) \) is odd, only the \( b_n \)'s will be nonzero (so start with a sine series).

What is not so obvious: the behavior of the \( a_n \)'s, \( b_n \)'s where there is (is not) a jump discontinuity.
Square Wave Example:

\[ f(x) = \begin{cases} 
-1 & -\pi < x < 0 \\
1 & 0 \leq x \leq \pi 
\end{cases} \]

\[ f(0) = 1 \text{ at } x=0; \text{ This tells us that the series converges to } \frac{f(0^-) + f(0^+)}{2} = 0. \]

The function is odd, so need only the \( b_n \)'s.

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x = \sum_{n=1}^{\infty} b_n \sin nx \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} -\sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx \]
\[
\frac{1}{\pi} \cos \frac{x \pi}{n} \bigg|_0^\pi + \frac{-\cos \left(\frac{\pi}{n}\right)}{\pi n} \bigg|_0^\pi \\
= \frac{1}{\pi} \left( \frac{1}{n} - \cos \left(\frac{\pi}{n}\right) \right) - \frac{1}{\pi} \left( \frac{\cos \left(\frac{\pi}{n}\right)}{n} - \frac{1}{n} \right) \\
= \frac{1}{\pi} \left[ \frac{1}{n} + \frac{1}{n} - \cos \left(\frac{\pi}{n}\right) - \cos \left(\frac{\pi}{n}\right) \right] = \frac{2}{\pi} \left[ \frac{1}{n} - \frac{\cos \left(\frac{\pi}{n}\right)}{n} \right] \\
= \begin{cases} 
\frac{2}{\pi} & n \text{ even} \\
\frac{4}{\pi n} & n \text{ odd} 
\end{cases}
\]

\[ f(x) \sum_{n=1}^{\infty} \frac{\sin \left(\frac{\pi x}{n}\right)}{n} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin \left(\pi x \frac{2k-1}{n}\right)}{2k-1} \]

* See picture of the partial sum.

** Coefficients that behave like \( \frac{1}{n} \) is the slowest converge associated with a jump discontinuity of the function, even at the endpoints -1 or 1.
Gibbs phenomenon for square wave

\[ f(x) = \frac{4}{\pi} \sum_{n=1}^{N} \frac{\sin((2n-1)x)}{(2n-1)x} \]
The Gibb's Phenomenon: For a finite number of terms, there is an overshoot of approximately 9% of the jump discontinuity (15% of the half-jump) of the overshoot moves closer to the point of discontinuity as N increases (N is the number of terms), but does not disappear.

Example 2: We also expect a behavior of the \( b_n \)'s and a Gibb's phenomenon for

\[
F(x) = \begin{cases} 
  x & -1 \leq x \leq 1 
\end{cases}
\]

What really matters is the periodic extension!
\[ b_n = \frac{1}{L} \int_{-L}^{L} x \sin \frac{n\pi x}{L} \, dx \quad \text{let } L = \pi. \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \quad \text{integrate by parts.} \]

\[ u = x \quad \frac{dv}{dx} = \sin nx \]
\[ \frac{du}{dx} = dx \quad v = -\cos nx \]

\[ = \frac{1}{\pi} \left[ \left. -\frac{x \cos nx}{n} \right|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n} \, dx \right]. \]

\[ = \frac{1}{\pi} \left[ \frac{-\cos n\pi - \cos (-n\pi)}{n} + \int_{-\pi}^{\pi} \frac{\cos nx}{n} \, dx \right]. \]

\[ = \frac{\cos n\pi - \cos n\pi}{\pi n} + \frac{1}{\pi} \left[ 0 - 0 \right] \]

\[ = \frac{2 \cos n\pi}{\pi n} \]

\[ f(x) = \sum_{n=1}^{\infty} \frac{2 \cos n\pi}{\pi n} \sin nx. \]

coefficients again behave like \( \frac{1}{n} \).

Gibbs phenomenon at \( x = -L \) \quad \text{and} \quad x = L. \]