[Lecture 322 Lecture 13]

Recap: For $f(x)$ piecewise smooth in $[-L, L]$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

* If the periodic extension of $f(x)$ has any jump discontinuities, then the a_n's and/or b_n's will have $\frac{1}{n}$ behavior (slow convergence of the series)

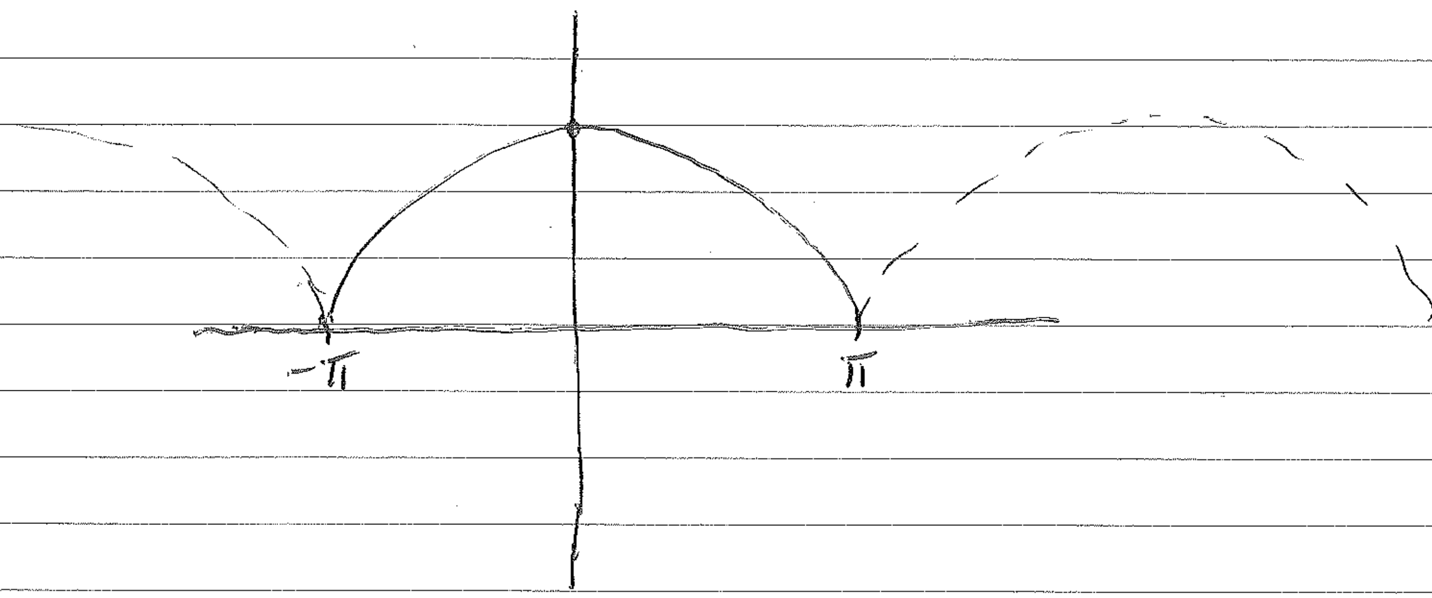
* At a point of discontinuity x_0, the series will converge to $\dfrac{f(x_0^-) + f(x_0^+)}{2}$

and it does not matter what the actual value of $f(x_0)$ is defined to be

* There will be a Gibb's phenomenon for finite sums at every point where the periodic extension of $f(x)$ has a jump

What is the behavior of the coefficients when the derivative $\frac{df(x)}{dx}$ has a jump discontinuity

Example: $f(x) = 1 - \left(\frac{x}{\pi}\right)^2 \qquad -\pi \leq x \leq \pi$



$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{x^2}{\pi^2}\right) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(1 - \frac{x^2}{\pi^2}\right) \cos nx \; dx$$

$$a_0 = \frac{1}{2\pi}\left[x\Big|_{-\pi}^{\pi} - \frac{x^3}{3\pi^2}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\pi + \pi - \left[\frac{\pi^3}{3\pi^2} - \frac{(-\pi)^3}{3\pi^2}\right]\right]$$

$$= \frac{1}{2\pi}\left[2\pi - \frac{2\pi^3}{3\pi^2}\right] = \frac{2\pi}{2\pi}\left[1 - \frac{1}{3}\right] = 2\pi/3$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(1 - \frac{x^2}{\pi^2}\right)\cos nx \, dx$$

$$\int_{-\pi}^{\pi} \cos nx \, dx = \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

integration by parts twice

$u = x^2$ $\quad$ $dv = \cos nx \, dx$

$du = 2x dx$ $\quad$ $v = +\frac{\sin nx}{n}$

$$= x^2\frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{\sin nx}{n} \, dx$$

$u = -2x$ $\quad$ $dv = \frac{\sin nx \, dx}{n}$

$du = -2dx$ $\quad$ $v = -\frac{\cos nx}{n^2}$

$$= \frac{2x\cos nx}{n^2}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2\cos nx}{n^2} \, dx$$

$$= \frac{2\pi \cos \pi n}{n^2} + \frac{2\pi \cos(-\pi n)}{n^2} - \frac{2 \sin nx}{n^3} \Big|_{-\pi}^{\pi}$$

$$= \frac{4\pi \cos n\pi}{n^2}$$

$$a_n = -\frac{1}{\pi^3} \frac{4\pi}{n^2} \cos n\pi = \frac{-4}{n^2\pi^2} \cos n\pi$$

$$1 - \left(\frac{x}{\pi}\right)^2 = \frac{2\pi}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi \cos nx$$

$\frac{1}{n^2}$ behavior as anticipated

$$F(x) = F_{even}(x) + F_{odd}(x)$$

$$= \frac{1}{2}\left[F(x) + F(-x)\right] + \frac{1}{2}\left[F(x) - F(-x)\right]$$

$F_{even}(x)$ may be represented by cosine series

$F_{odd}(x)$ " " " " sine series

Conclusion: We can only technically write the "=" if there are no jump discontinuities anywhere in the periodic <u>extension</u> of $f(x)$

Then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

Otherwise write " $\sim$ "

$f(x) \sim \ldots$

because at some points the series converges to something else.

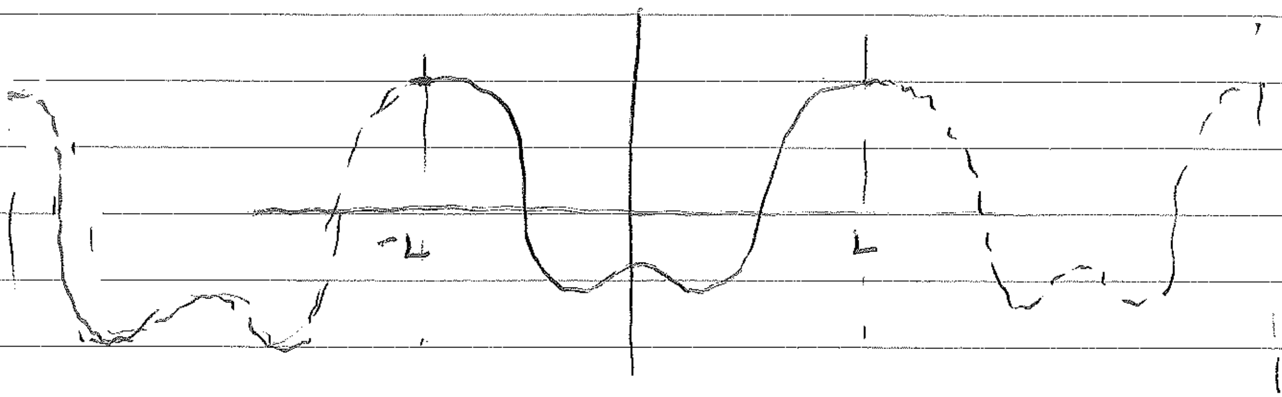
Next: Formal statements regarding

* Convergence
* Differentiation term by term
* Integration term by term

Statements Assume $f(x)$ piecewise smooth in $[-L, L]$

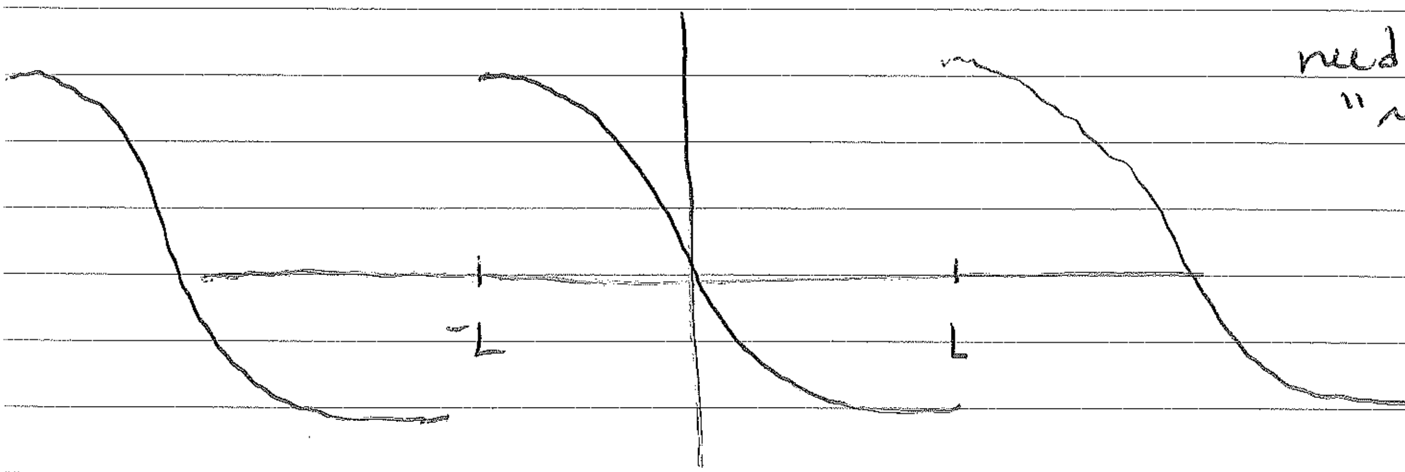
For $f(x)$ continuous, the Fourier series for $f(x)$ converges to $f(x)$ on $-L \le x \le L$ only if $f(-L) = f(L)$

{ otherwise there is a jump discontinuity at $x_0 = -L$, $x_0 = L$ leading to $\frac{1}{2}$ behavior, with convergence to the average function value at $x_0 = -L$, $x_0 = L$ }



"=" ok

vs



need "~"

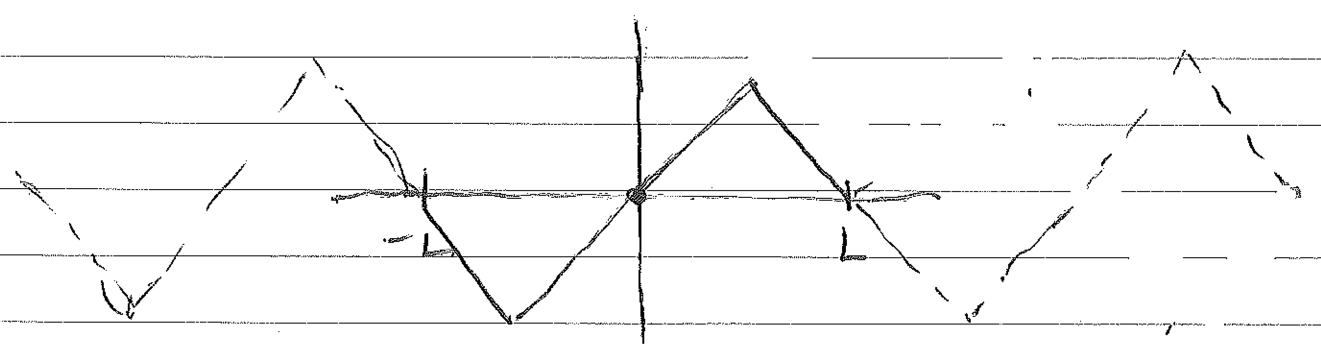
What about sines and cosines separately?

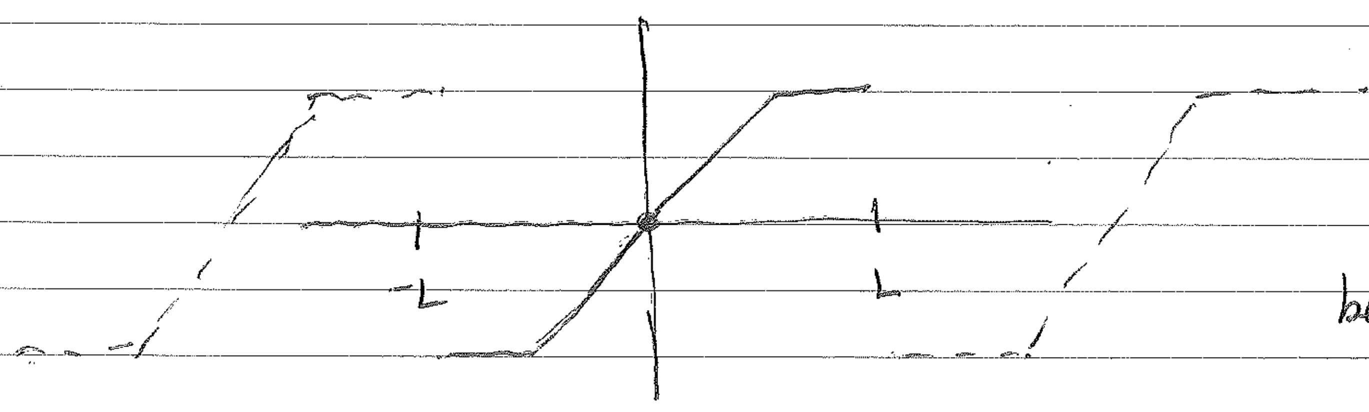
If $F_e(x)$ even $\Rightarrow$ function automatically satisfies $F(-L) = F(L)$

$\Rightarrow$ no discontinuity in the function at the endpoints; possibly discontinuity in the slope at the endpoints

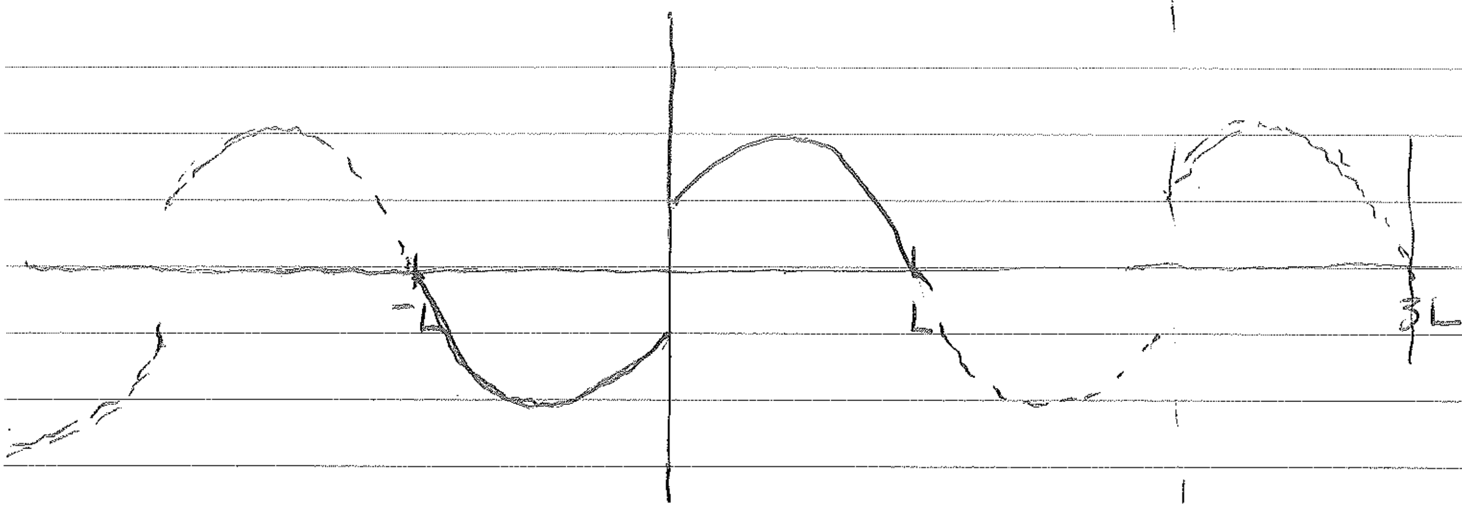
If $F_o(x)$ odd $\Rightarrow$ to avoid discontinuity in the function at the endpoints, need $F(-L) = F(L) = 0$

{ $F(0) = 0$ also for continuity inside $-L \le x \le L$ }





$\frac{1}{n}$ behavior



$\frac{1}{n}$ behavior

If working in $0 \le x \le L$ $\Rightarrow$

$f(0) = f(L) = 0$