For steady flow over an infinite flat plate, we found a 1D-1C solution

\[ u = u(y, t) \hat{x} \]

\[ \frac{u}{u_0} = 1 - \frac{1}{\sqrt{\pi}} \int_0^1 \exp \left( -\frac{s^2}{4t} \right) \, ds \]
\[ \eta = \frac{y}{(y^2 + 1)^{1/2}} \]

Corresponding to the plate impulsively set in motion at \( t = 0 \)

\[ u(y, 0) = 0, \quad y > 0 \]
\[ u(0, t) = u_0, \quad t > 0 \]
\[ u(y, t) = 0, \quad y \to \infty, \quad t > 0 \]

This solution automatically satisfies conservation of mass \( \frac{\partial u}{\partial x} = 0 \) and we used momentum eqn. only
Now consider uniform flow impinging on a finite plate.

There is a region of rapid change (width \( s \)) connected to a slowly varying "after solution" (in this case constant velocity).

Notice now that there is variation in both \( x \) and \( y \): 
\[
u(x,y) \quad \frac{d}{dx} \Rightarrow \]
\[
v(x,y) \quad \frac{d}{dy} \Rightarrow \]

must be a \( v(x,y) \) to conserve mass:

\[
\frac{d u(x,y)}{d x} + \frac{d v(x,y)}{d y} = 0
\]

So the simplest theory will be \( zD - zC \).
In general, a "boundary layer" is a region of rapid change (or "inner solution") connected to a slowly varying "outer solution."

We must find "inner" and "outer" solutions and match them in the middle.

We want to construct a uniformly valid global solution

\[ u(x,t) = u^{in}(x,t) + u^{out}(x,t) - u^{\text{match}}(x,t) + \text{error} \]

See Bender and Orszag Ch 9
(ODE !)

Acheson example. \( \varepsilon u'' + u' = 1 \) \( u = u(y) \)
\[ u(0) = 0 \quad u(1) = 2 \]
\( \varepsilon \to 0^+ \)

This is a singular perturbation problem with a small parameter in front of the highest derivative. The problem with \( \varepsilon = 0 \) is not smoothly connected to the problem for \( \varepsilon \to 0^+ \), and in fact the former has no solution.

\( \varepsilon = 0 \) \( u' = 1 \) \( \Rightarrow u = y + c \)
\[ u(0) = 0 \Rightarrow c = 0 \]
\[ u(1) = 2 \Rightarrow 2 = 1 \] Not true.

So the unperturbed problem cannot satisfy the boundary conditions.

On the other hand, the full problem has an exact solution

\[ u(y) = y + \left( \frac{1 - \exp \left( -y/\varepsilon \right)}{1 - \exp \left( -\frac{1}{3} \right)} \right) \]
as can be found by direct integration
\[ \varepsilon u'' + u' = 0 \Rightarrow u_h(y) = A \exp \left( -\frac{y}{\varepsilon} \right) + B \]

\[ u_p(y) = y \Rightarrow u(y) = A \exp \left( -\frac{y}{\varepsilon} \right) + B + y \]

\[ u(0) = 0 \Rightarrow A + B = 0 \]
\[ u(1) = 2 \Rightarrow A \exp \left( -\frac{1}{\varepsilon} \right) + B = 1 \]
\[ \Rightarrow B \left( 1 - \exp \left( -\frac{1}{\varepsilon} \right) \right) = 1 \]

\[ B = \frac{1}{1 - \exp \left( -\frac{1}{\varepsilon} \right)} \quad , \quad A = -\frac{1}{1 - \exp \left( -\frac{1}{\varepsilon} \right)} \]

\[ u(y) = y + \frac{\exp \left( -\frac{y}{\varepsilon} \right)}{\left( 1 - \exp \left( -\frac{1}{\varepsilon} \right) \right)} + \frac{1}{\left( 1 - \exp \left( -\frac{1}{\varepsilon} \right) \right)} \]

\[ = y + \frac{1 - \exp \left( -\frac{y}{\varepsilon} \right)}{\left( 1 - \exp \left( -\frac{1}{\varepsilon} \right) \right)} \]
If we solve this perturbatively, let

\[ u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \]

Plug in (dropping superscript zero for new)

\[ \varepsilon \left( u_0 + \varepsilon u_1 + \cdots \right)'' + \left( u_0 + \varepsilon u_1 + \cdots \right)' = 1 \]

\[ \varepsilon \to 0^+ \implies u_0' \sim 1 \]

and for the outer solution we should use the boundary condition away from the boundary layer near \( y = 0 \) \( \implies \)

\[ u_0' (y) \sim 1 \quad u_0 (1) = 2 \]

\[ u_0 = y + c, \quad u_0 (1) = 2 = 1 + c \implies c = 1 \]

\[ u^\text{out} (y) = y + 1, \quad u^\text{out} (y) = O(y+1 + o(\varepsilon)) \]

We are treating the outer solution as inviscid (away from the wall)
For the inner solution, we need to find an approximation valid near $y=0$ in the region of rapid change.

To "zoom in" near $y=0$, let $s \equiv \frac{y}{\varepsilon^p}$

where we presume that the thickness of the boundary layer scales as a power of $\varepsilon$ and we need to find the power $p$.

$$\frac{du}{dy} = \frac{du}{ds} \frac{ds}{dy} = \frac{s^p}{E^p} \frac{du}{ds} \quad \frac{d^2 u}{dy^2} = \frac{1}{E^p} \frac{d^2 u}{ds^2}$$

and our equation is

$$\frac{E}{E^p} \frac{d^2 u}{ds^2} + \frac{1}{E^p} \frac{du}{ds} = 1$$

Now let $u^m(s) = u_0(s) + \varepsilon u_1(s) + \cdots$

Plug in $\Rightarrow$ drop "in". For now

$$E^{1-2p} \left[ u_0 + \varepsilon u_1 + \cdots \right]'' + E^{-p} \left[ u_0 + \varepsilon u_1 + \cdots \right]' = 1$$

and we can find a balance of large terms if we choose

(new primes are derivatives wrt $s$)
\[ p = 1 \Rightarrow \frac{1}{\varepsilon} \frac{d^2 u_0}{d \xi^2} \sim -\frac{1}{\varepsilon} \frac{d u_0}{d \xi} \]

and we've dropped terms \( O(1) \) and smaller.

We have the inner boundary condition \( u(0) = 0 \)

\[ \Rightarrow u_0^m(\xi) = A \left(1 - \exp\left(-\frac{\xi}{\varepsilon}\right)\right) \]

\[ u_0^m(\xi) = A \left(1 - \exp\left(-\frac{s}{\varepsilon}\right)\right) + O(\varepsilon) \]

We still have one coefficient \( A \) but we need to match in the middle.

\[ \lim_{y \to 0} u_0^{out}(y) = \lim_{\xi \to \infty} u_0^m(y) \]

where \( \xi \to \infty \Rightarrow y = O(1) \), \( \varepsilon \to 0^+ \)

\[ \lim_{y \to 0} u_0^{out} + O(\varepsilon) = \lim_{\xi \to \infty} u_0^m + O(\varepsilon) \]

and at lowest order we have

\[ \lim_{y \to 0} y + 1 = \lim_{\xi \to \infty} A \left(1 - \exp\left(-\frac{s}{\varepsilon}\right)\right) \]
\[ u = u_{\text{out}} + u_{\text{match}} \]

So finally

\[ u \sim u_{\text{out}} + u_{\text{match}} + o(\varepsilon) \]

\[ u(y) \sim y + 1 + (1 - \exp(-\frac{y}{\varepsilon})) - 1 + o(\varepsilon) \]

\[ u(y) \sim y + (1 - \exp(-\frac{y}{\varepsilon})) + o(\varepsilon) \]

Compare to the exact soln.

\[ u(y) = y + \frac{(1 - \exp(-\frac{y}{\varepsilon}))}{(1 - \exp(-\frac{1}{\varepsilon}))} \]

\[ = y + (1 - \exp(-\frac{y}{\varepsilon})) \left\{ 1 + \exp\left(-\frac{1}{\varepsilon}\right) + \cdots \right\} \]

\[ \text{satisfies } u(0) = 0 \quad \checkmark \]

\[ \text{satisfies } u(1) = 2 - \exp(-\frac{1}{\varepsilon}) \]