Why can we differentiate a Fourier series term by term?

If we write

\[ F(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n \pi x}{L} + b_n \sin \frac{n \pi x}{L} \right) \]

when will the true derivative \( F'(x) \) be given by

\[ F'(x) \sim \sum_{n=1}^{\infty} \left\{ \frac{n \pi}{L} a_n \sin \frac{n \pi x}{L} + \left( \frac{n \pi}{L} \right) b_n \cos \frac{n \pi x}{L} \right\} \]

We need the same conditions as for convergence of the series to \( F(x) \):

** F(x) continuous in \([-L, L]\)

** F(-L) = F(L)

\[ \text{[Cosine series]} \quad F(-L) = F(L) \text{ satisfied automatically} \]

\[ \text{[Sine series]} \quad \text{we also need } F(-L) = F(L) = 0 \]

\[ \sum \text{ and } F(0) = 0 \text{ if given in } [0, L] \]
i.e. we do not allow any jump discontinuities in the periodic extension of $f(x)$.
Various things to show:

1. If the conditions are violated, show that term by term differentiation may lead to the "wrong" result. (back)

2. If all restrictions are satisfied, how do we prove that we can differentiate term by term? (back)

3. If \( f(x) \) has a jump discontinuity, how do we find an appropriate series for \( f'(x) \) from the series for \( f(x) \)? (homework)
[Cosine series] \( f(-L) = f(L) \) automatically...

[Sine series] We also need \( f(-L) = f(L) = 0 \)
\[ \text{and } f(0) = 0 \text{ if given } f(x) \text{ in } [0, L] \]

Book sometimes gives the function in \([0, L]\)
and asks about the cosine series or sine series.

[Example 1] Consider the sine series of

\[ f(x) = x \text{ in } [0, L] \]

0 Make an odd extension in \([-L, 0]\), then
a periodic extension outside \([-L, L]\)
Conclusion: we cannot differentiate term by term because \( F(L) \neq F(-L) \neq 0 \)

What happens if we do it anyway?

\[ F(x) = x \text{ in } [-\pi, \pi] \]

The true \( F'(x) = 1 \)

\[ F(x) \sim \frac{1}{\pi} \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx \]

\[ \Rightarrow F(x) \sim 2 \sum_{n=1}^\infty \frac{1}{n} (-1)^{n+1} \sin nx \]

Take the term by term derivative \( \Rightarrow \)

\[ 2 \sum_{n=1}^\infty (-1)^{n+1} \cos nx \]

\[ = 2 \left\{ \cos x - \cos 2x + \cos 3x + \ldots \right\} \]

What is the cosine series of \( F'(x) = 1 \)

\[ 1 = a_0 + \sum_{n=1}^\infty a_n \cos nx = a_0 + \sum_{n=1}^\infty a_n \cos nx \]

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \, dx = 1 \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \, dx = \frac{1}{n\pi} \sin nx \int_{-\pi}^{\pi} = 0 \]
If all the conditions are satisfied, how do we prove that we can differentiate term by term?

E.g., consider a sine series

Given \( f(x) \) in \( 0 \leq x \leq 1 \), make an odd extension in \( [-1,0] \), a periodic extension outside \( [-1,1] \).

We need

* \( f(x) \) continuous

* \( f(x) \) may have finite jump discontinuities

* \( f(0) = f(1) = 0 \)

Write

\[
\begin{align*}
S(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \\
F(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}
\end{align*}
\]

We need to show

\[
\sum_{n=1}^{\infty} \frac{n\pi}{L} b_n \cos \frac{n\pi x}{L} = a_0 + \sum_{n=1}^{\infty} \frac{n\pi}{L} a_n \cos \frac{n\pi x}{L}
\]
or \[ a_0 = 0 \]

\[ \frac{n \pi}{L} b_n = a_n \quad n \neq 0 \]

Show \[ a_0 = 0 \]

\[ a_0 = \frac{1}{a L} \int_{-L}^{L} f'(x) \, dx = \frac{1}{L} \int_{0}^{L} f'(x) \, dx \]

\[ = \frac{1}{L} \left. f(x) \right|_{0}^{L} = \frac{1}{L} \left[ f(L) - f(0) \right] \]

\[ = 0 \quad f(L) = f(0) \quad \text{need this restriction} \]

Show \[ a_n = \frac{n \pi}{L} b_n \quad n \neq 0 \]

\[ a_n = \frac{1}{L} \int_{-L}^{L} f'(x) \cos \frac{n \pi x}{L} \, dx \quad u = \cos \frac{n \pi x}{L} \quad dv = f'(x) \, dx \]

\[ du = \frac{n \pi}{L} \sin \frac{n \pi x}{L} \quad V = f(x) \]

\[ = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n \pi x}{L} \, dx \]

\[ = -\frac{2}{L} \left. \cos \frac{n \pi x}{L} f(x) \right|_{0}^{L} + \frac{2}{L} \int_{0}^{L} \frac{n \pi}{L} \sin \frac{n \pi x}{L} f(x) \, dx \]
\[ a_n = \frac{2}{L} \int_0^L \frac{\sin(n\pi x)}{L} f(x) \, dx \]

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right) \]

\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \]

Comparing \( a_n \) and \( b_n \), we see

\[ a_n = \frac{n\pi}{L} b_n \]
Homework Problem: Let's assume we have a jump discontinuity in $f(x)$, e.g.,

$$f(x) = \begin{cases} -x & 0 < x < \frac{1}{2} \\ x - \frac{1}{2} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$\cos$ series

$\sin$ series

$$f'(x) = \begin{cases} -1 & 0 < x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$$
\[ F(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \]

\[ F'(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \]

but \( b_n \neq -\frac{n\pi}{L} a_n \)

Using integration by parts on \( b_n = \frac{2}{L} \int_{x_0^-}^{x_0^+} \frac{dF(x)}{dx} \sin \frac{n\pi x}{L} \, dx \) we find

\[ b_n = \frac{2}{L} \left[ F(x_0^-) - F(x_0^+) \right] \frac{\sin n\pi x_0}{L} - \frac{n\pi}{L} a_n \]

\[ b_n = \frac{2}{L} \left\{ F(x) \sin \frac{n\pi x}{L} \bigg|_{x_0^-}^{x_0^+} + F(x) \sin \frac{n\pi x}{L} \bigg|_{x_0^-}^{x_0^+} \right\} \]

\[ -\int_0^L F(x) \cos \frac{n\pi x}{L} \, dx \]