$\boxed{\text{Lecture 14 Math 322}}$

When can we differentiate a Fourier series term by term?

IF we write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

when will the true derivative $f'(x)$ be given by

$$f'(x) \sim \sum_{n=1}^{\infty} \left\{ \left(\frac{-n\pi}{L}\right) a_n \sin\frac{n\pi x}{L} + \left(\frac{n\pi}{L}\right) b_n \cos\frac{n\pi x}{L} \right\}$$

We need the same conditions as for convergence of the series to $f(x)$

** $f(x)$ continuous in $[-L, L]$

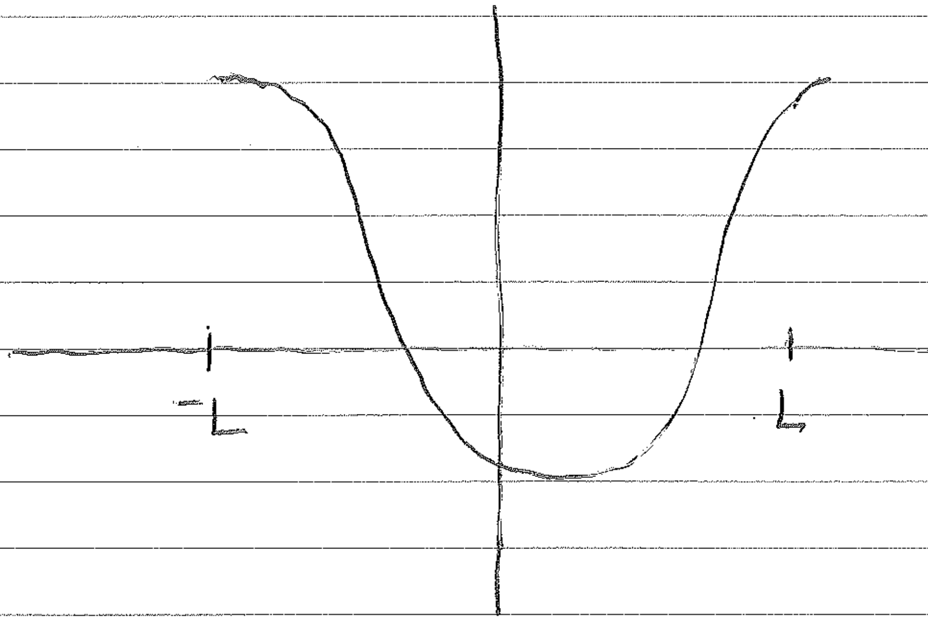
** $f(-L) = f(L)$

$\boxed{\text{cosine series}}$ $f(-L) = f(L)$ satisfied automatically

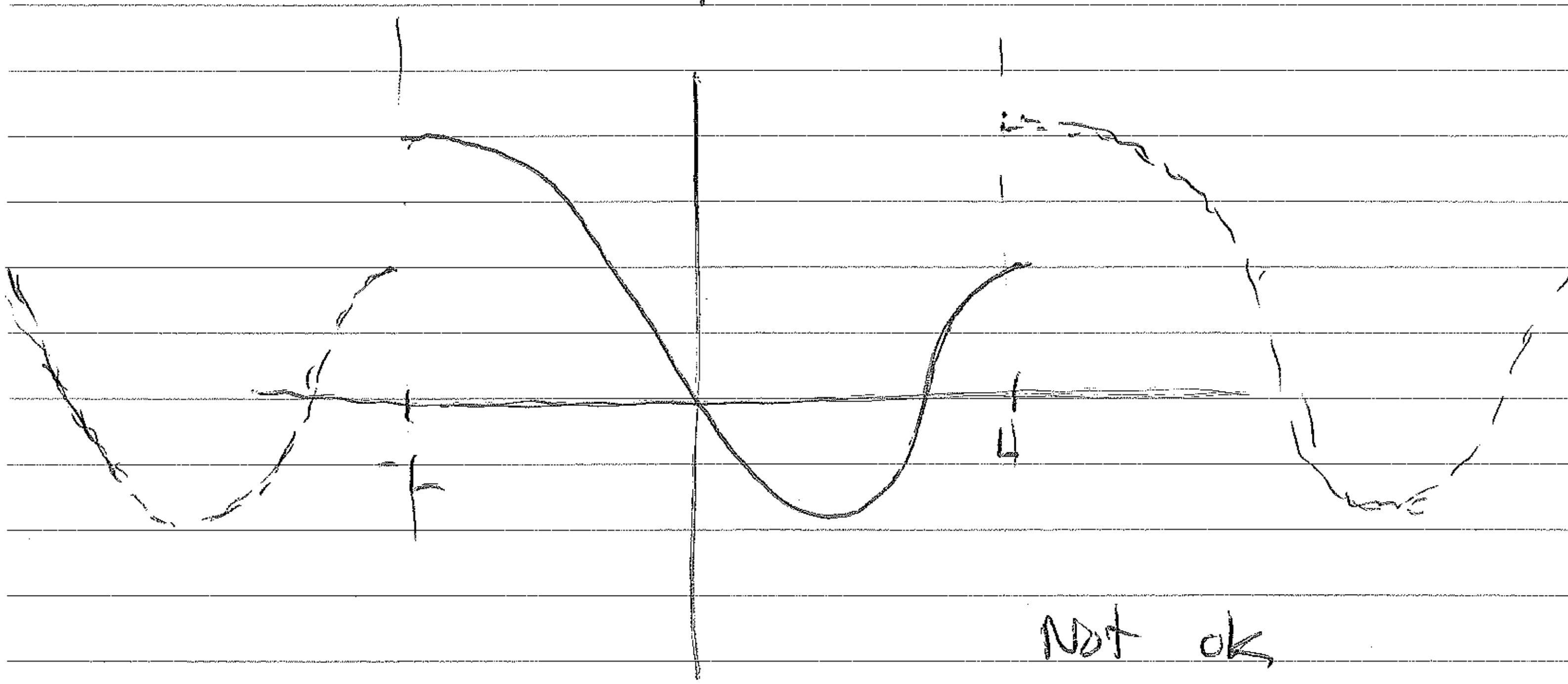
$\boxed{\text{sine series}}$ we also need $f(-L) = f(L) = 0$

{ and $f(0) = 0$ if given in $[0, L]$ }

i.e. we do not allow any jump discontinuities in the periodic extension of $F(x)$



ok to differentiate term by term



Not ok

Various things to show:

1. If the conditions are violated, show that term by term differentiation may lead to the "wrong" result.
(book)

2. If all restrictions are satisfied, how do we prove that we can differentiate term by term?
(book)

3. If $f(x)$ has a jump discontinuity, how do we find an appropriate series for $f'(x)$ from the series for $f(x)$?
(homework)

cosine series $f(-L) = f(L)$ automatically ...

Sine series We also need $f(-L) = f(L) = 0$

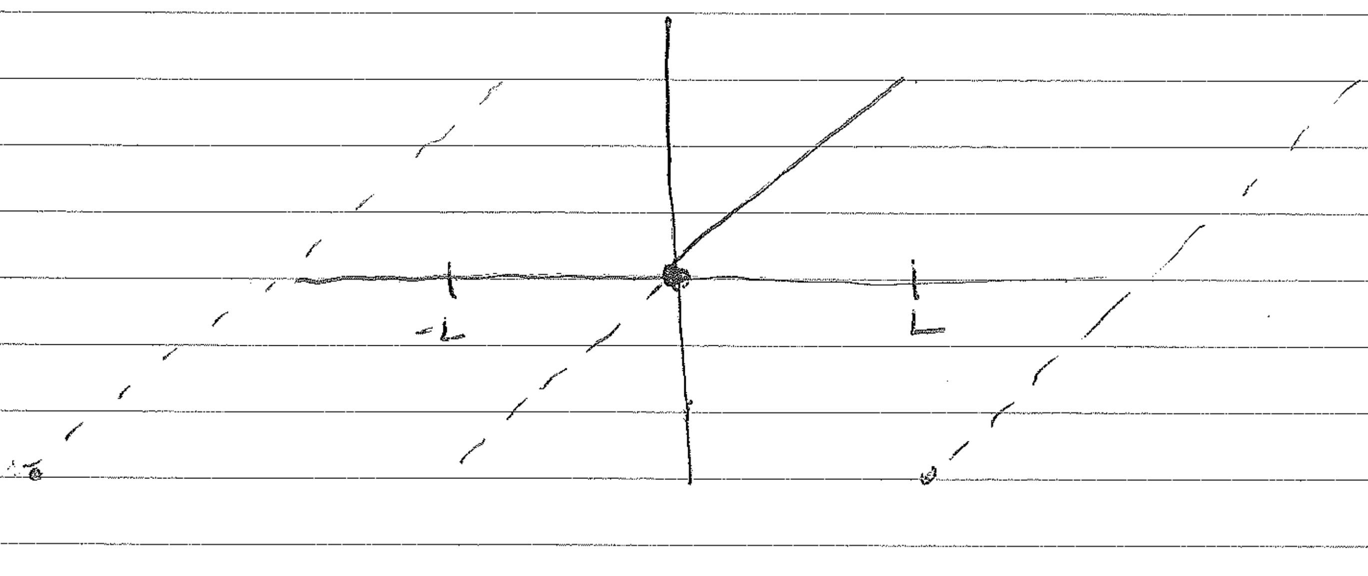
{and $f(0) = 0$ if given $f(x)$ in $[0, L]$}

Book sometimes gives the function in $[0, L]$ and asks about the cosine series or sine series.

Example 1 Consider the sine series of

$f(x) = x$ in $[0, L]$.

① Make an odd extension in $[-L, 0]$, then a periodic extension outside $[-L, L]$



Conclusion: we cannot differentiate term by term because $f(L) \neq f(-L) \neq 0$

What happens if we do it anyway?

* $f(x) = x$ in $[-\pi, \pi]$

The true $f'(x) = \underline{1}$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \quad , \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$\Rightarrow \quad f(x) \sim 2 \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} \sin nx$$

Take the term by term derivative $\Longrightarrow$

$$2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$$

$$= 2 \left\{ \cos x - \cos 2x + \cos 3x + \cdots \right\}$$

---

what is the cosine series of $f'(x) = \underline{1}$

$$\underline{1} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

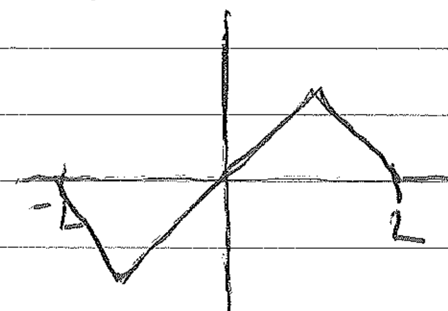
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \, dx = \underline{1} \quad , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \, dx$$

$$= \frac{1}{n\pi} \sin nx \Big|_{-\pi}^{\pi} = 0$$

If all the conditions are satisfied, how do we prove that we can differentiate term by term?

E.g. consider a sine series

Given $F(x)$ in $0 \le x \le L$; make an odd extension in $[-L, 0]$; a periodic extension outside $[-L, L]$



We need

* $F(x)$ continuous

* $F'(x)$ may have finite jump discontinuities

* $F(0) = F(L) = 0$

Write $F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$$F'(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

We need to show

$$\sum_{n=1}^{\infty} \frac{n\pi}{L} b_n \cos \frac{n\pi x}{L} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

or $\quad a_0 = 0$

$$\frac{n\pi}{L} b_n = a_n \qquad n \neq 0$$

Show $\quad a_0 = 0$:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{L}\int_0^L f'(x)\,dx$$

$$= \frac{1}{L} f(x)\Big|_0^L = \frac{1}{L}\left[f(L) - f(0)\right]$$

$$= 0 \quad \text{if} \quad f(L) = f(0)$$ need this restriction !

Show $\quad a_n = \frac{n\pi}{L} b_n \qquad n \neq 0$

$$a_n = \frac{1}{L}\int_{-L}^{L} f'(x) \cos\frac{n\pi x}{L}\,dx$$

$u = \cos\frac{n\pi x}{L} \qquad dv = f'(x)dx$

$du = -\frac{n\pi}{L}\sin\frac{n\pi x}{L} \qquad v = f(x)$

$$= \frac{2}{L}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\cos\frac{n\pi x}{L} f(x)\Big|_0^L + \frac{2}{L}\int_0^L \frac{n\pi}{L}\sin\frac{n\pi x}{L} f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L \frac{n\pi}{L} \sin\frac{n\pi x}{L}\, f(x)\, dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

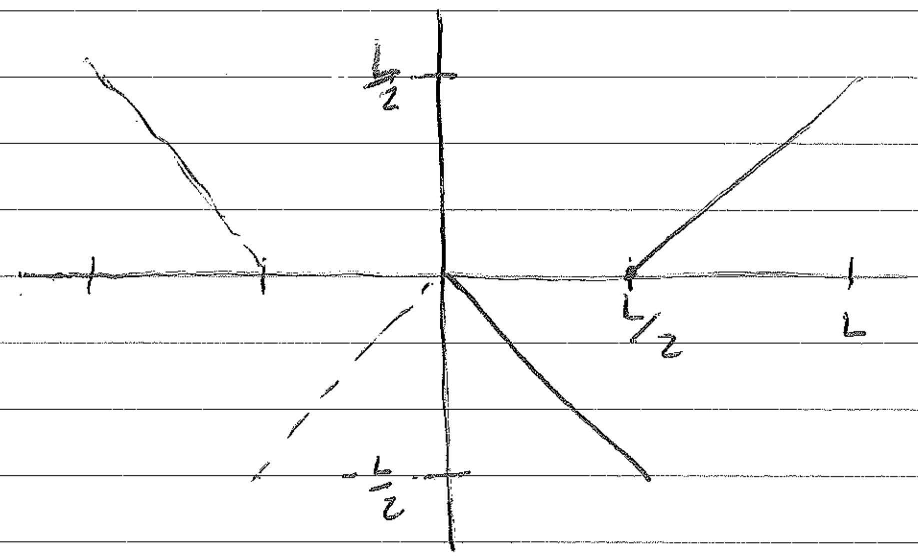
$$\Longrightarrow \quad b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

Comparing a_n and b_n, we see

$$a_n = \frac{n\pi}{L} b_n$$

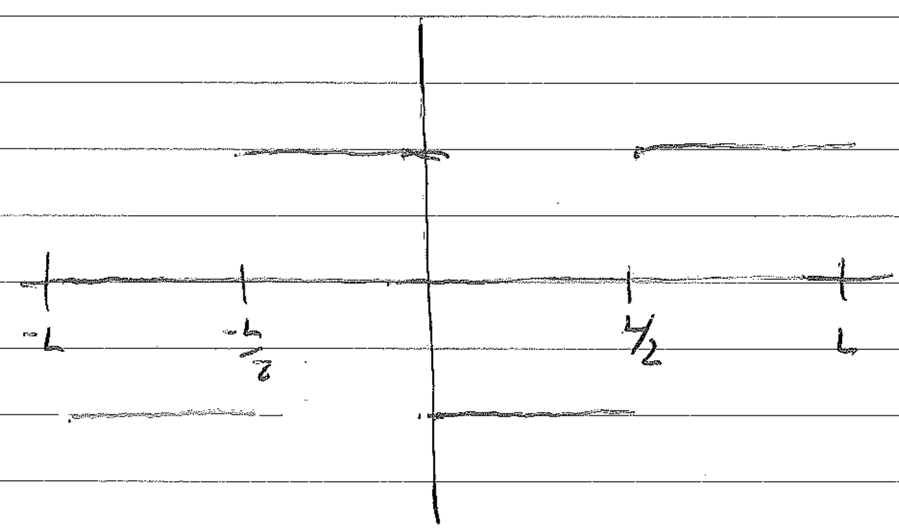
$\boxed{\text{Homework Problem}}$ Lets assume we have a jump discontinuity in $F(x)$, e.g.

$$F(x) = \begin{cases} -x & 0 < x < \frac{L}{2} \\ x - \frac{L}{2} & \frac{L}{2} \le x \le L \end{cases}$$



cosine series

$$F'(x) = \begin{cases} -1 & 0 < x < L/2 \\ 1 & \frac{L}{2} \le x \le L \end{cases}$$



sine series

$$F(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$F'(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

but $\quad b_n \neq -\frac{n\pi}{L} a_n \quad$!!!

Using integration by parts on

$$b_n = \frac{2}{L} \int_0^L \frac{dF(x)}{dx} \sin \frac{n\pi x}{L}\, dx$$ we find

$$b_n = \frac{2}{L} \left[F(x_0^-) - F(x_0^+) \right] \sin \frac{n\pi x_0}{L} - \frac{n\pi}{L} a_n$$

$$b_n = \frac{2}{L} \left\{ F(x) \sin \frac{n\pi x}{L} \Big|_0^{x_0^-} + F(x) \sin \frac{n\pi x}{L} \Big|_{x_0^+}^{L} - \int_0^L \frac{n\pi}{L} F(x) \cos \frac{n\pi x}{L}\, dx \right\} \ldots$$