The Merits of Non-dimensionalization

Consider inviscid flow over a sphere:

\[ \text{Fluid } p, \mu \]

\[ \rho \quad \overrightarrow{U_0} \quad x \]

The dependent flow quantities at a point \( x \) in the flow depend on

\[ p = p(x, \rho, \mu, \rho_0, U_0, a) \]

\[ u = u(x, \rho, \mu, \rho_0, U_0, a) \]

If we non-dimensionalize:

\[ \frac{p}{(\rho_0 U_0^2)}, \frac{u}{U_0}, \frac{x}{a}, \frac{\rho_0}{\rho U_0^2} \]

we have not used \( \mu \) so there needs to be one more group \( \Rightarrow \frac{U_0 \rho_0 p}{\mu} = Re \)

\[ \frac{p}{\rho U_0^2} = f \left( \frac{x}{a}, \frac{\rho_0}{\rho U_0^2}, \frac{U_0 \rho_0}{\mu} \right) \]

\[ \frac{u}{U_0} = g \left( \frac{x}{a}, \frac{\rho_0}{\rho U_0^2}, \frac{U_0 \rho_0}{\mu} \right) \]
Math 705

Back to flow over the flat plate

\[ \mathbf{u}_0 \hat{x} \]

\[ (0,0) \]

Outer solution is \( \mathbf{u} = \mathbf{u}_0 \hat{x} \)

We want to derive equations for the inner solution where the velocity is changing rapidly in the \( \hat{y} \)-direction when \( \hat{y} \to 0 \).

Start with 2D-AC incompressible flow (steady)

\[
\left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) u = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

\[
\left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) v = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]
1. Rescale the equations appropriately to "zoom in" near the boundary layer.

2. Look for a similarity solution that converts the \( \hat{x} \)-momentum PDE to an ODE

\[
\eta = \frac{y}{y(x)}
\]

where the thickness of boundary layer \( y(x) \) is allowed to change in \( x \).

3. Calculate the drag resulting from viscosity.
change in $\hat{x}$- direction is "slow" $\Rightarrow \frac{\partial}{\partial x} = O\left(\frac{1}{L}\right)$

change in $\hat{y}$- direction is "fast" $\Rightarrow \frac{\partial}{\partial y} = O\left(\frac{1}{S}\right)$

$S << 1$

Non-dimensionalize the equations as follows:

$x' = \frac{x}{L}, \quad y' = \frac{y}{S}$

$u' = u' = \frac{u}{u_0}, \quad v' = \frac{v}{u_0 S/L}, \quad p' = \frac{p}{\rho u_0^2}$

This is an ansatz!

Let's think about the terms in $\hat{x}$-momentum:

$u' \frac{du'}{dx} \Rightarrow u_0 u' \frac{u_0}{L} \frac{du'}{dx} = \frac{u_0^2}{L} u' \frac{du'}{dx}$

$v' \frac{du'}{dy} \Rightarrow u_0 \frac{v'}{S} \frac{u_0}{L} \frac{du'}{dy} = \frac{u_0^2}{L} v' \frac{du'}{dy}$

$-\frac{1}{\rho} \frac{dp'}{dx} \Rightarrow -\frac{1}{\rho} \frac{u_0^2}{L} \frac{dp'}{dx} = -\frac{u_0^2}{L} \frac{dp'}{dx}$
\[
\sqrt{2} \frac{d^2 u}{dx^2} \rightarrow \sqrt{\frac{1}{2}} \frac{d^2}{dx^1} \quad u_0 u' = \frac{\sqrt{2} u_0}{L^2} \frac{d^2 u'}{dx^1^2}
\]

\[
\frac{d^2 u}{dy^2} \rightarrow \sqrt{\frac{1}{2}} \frac{d^2}{dy^1} \quad u' = \frac{2 u_0}{\delta^2} \frac{d^2 u'}{dy^1^2}
\]

Divide by \( \frac{u_0}{L} \):

\[
\frac{u'}{dx^1} + \frac{v}{\delta} \frac{du'}{dy^1} = -\frac{dp'}{dx^1}
\]

\[
+ \frac{2 u_0}{L} \frac{1}{u_0^2} \frac{d^2 u'}{dx^1^2} + \frac{v u_0}{\delta^2} \frac{1}{u_0} \frac{d^2 u'}{dy^1^2}
\]

\[\uparrow\]

\[
\sqrt{\frac{2}{L u_0}} \quad \frac{v}{\delta} \frac{1}{8 u_0}
\]

Now let we take \( v \to 0, \quad u_0 = O(1), \quad L = O(1) \)

It is clear that \( \frac{d^2}{L u_0} \ll \frac{1}{L u_0} \frac{L^2}{\delta^2} \)

For \( S \ll L \)
and we take \( \frac{\nu^2}{L^4} \frac{L^2}{\delta^2} = O(1) \Rightarrow \)

\[ \delta^2 \sim \frac{\nu L}{u_0 L} = \frac{\nu}{u_0} \frac{L^2}{Re} \]

\[ \Rightarrow \delta \sim \frac{L}{\sqrt{Re}} \]

So this will be our presumed scaling with a balance of terms:

nonlinear term \( \sim \) pressure + \( \nu \frac{\partial^2 u}{\partial y^2} \)

and we drop \( \nu \frac{\partial^2 u}{\partial x^2} \) for \( \text{Re} \to \infty \)
what about j -momentum? let's do a few terms

\[
\frac{dx}{\partial t} = \frac{u_0 u}{L} \frac{1}{L} \frac{dv}{\partial x} = \frac{u_0^2}{L} \frac{1}{L} \frac{u' dv}{\partial x}
\]

\[
-\frac{1}{S} \frac{dp}{\partial y} = -\frac{1}{S} \rho u_0^2 \frac{dp'}{\partial y} = -\frac{u_0^2}{S} \frac{dp'}{\partial y}
\]

\[
\frac{udv}{\partial y} \Rightarrow (\frac{u_0 \sigma}{L}) \frac{1}{S} \frac{v' dv'}{\partial y} = \frac{u_0^2}{L^2} \frac{f}{L} \frac{v' dv'}{\partial y}
\]

\[
\frac{dv}{\partial x^2} \Rightarrow \frac{1}{L^2} \frac{d^2}{\partial x^2} \frac{u_0 \sigma}{L} v' = \frac{u_0 \sigma}{L^2} \frac{d^2 v'}{\partial x^2}
\]

\[
\frac{v}{dy^2} \Rightarrow \frac{1}{S^2} \frac{d^2}{\partial y^2} \frac{u_0 \sigma}{L} v' = \frac{u_0 \sigma}{S^2} \frac{d^2 v'}{\partial y^2}
\]

So now we could divide by \( \frac{u_0^2 f}{L} \) ⇒

\[
\frac{u' dv'}{\partial x} + \frac{v' dv'}{\partial y} = -\frac{u_0^2}{S} \frac{L}{L^2} \frac{dp'}{L} \frac{1}{S} \frac{dp'}{\partial y}
\]

\[
+ \frac{u_0 \sigma}{L^2} \frac{f}{L} \frac{L}{S} \frac{d^2 v'}{\partial x^2} + \frac{u_0 \sigma}{S^2} \frac{L}{L} \frac{d^2 v'}{\partial y^2}
\]
\[
\frac{u'}{dx'} + v' \frac{dv'}{dy'} = -\frac{L^2}{S^2} \frac{dp'}{dy'}
\]

\[
+ \frac{L}{\nu_0 L} \frac{d^2 v'}{dx'^2} + \frac{2}{\nu_0 L} \frac{d^2 v'}{dy'^2}
\]

\[
\frac{u'}{dx'} + v' \frac{dv'}{dy'} = -\frac{L^2}{S^2} \frac{dp'}{dy'} + \frac{1}{Re} \frac{d^2 v'}{dx'^2} + \frac{1}{Re} \frac{d^2 v'}{dy'^2}
\]

with \( \frac{S}{L} \ll 1 \), \( Re \to \infty \)

\[
\Rightarrow \frac{dp'}{dy'} \approx 0
\]

Mass conservation

\[
\frac{du}{dx} = \frac{U_0}{L} \quad \frac{du'}{dx'} \quad \frac{dv}{dy} = \frac{U_0 S}{L} \frac{dv'}{dy'}
\]

Same order of magnitude \( \Rightarrow \)

\[
\frac{du'}{dx'} + \frac{dv'}{dy'} = 0
\]
So our system reduces to

\[
\frac{u'}{dx} + v'\frac{du'}{dy} = -\frac{dp'}{dx} + \frac{d^2u'}{dy'^2}
\]

\[\frac{dp'}{dy'} = 0 \quad \text{and} \quad \frac{du'}{dx} + \frac{dv'}{dy} = 0\]

or back to the dimensional equations

\[
\frac{u}{dx} + v\frac{du}{dy} = -\frac{1}{e} \frac{dp}{dx} + v \frac{d^2u}{dy^2}
\]

\[\frac{dp}{dy} = 0 \quad \text{and} \quad \frac{du}{dx} + \frac{dv}{dy} = 0\]

which is 2 equations for 3 unknowns \(u, v, p\)!

So let's assume that the pressure gradient \(\frac{dp}{dx}\) is determined by the \underline{outer inviscid flow} (and will be a driver for this inner flow).

But if the outer flow is constant,

\[
ue \frac{du}{dx} = -\frac{1}{e} \frac{dp}{dx} = 0
\]

(revisit later for \(ue(x)\) non-constant)
We arrive at

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \equiv \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

\[ \text{Re} = \frac{U_0 L}{v} \to 0, \quad \delta = \frac{L}{\sqrt{\text{Re}}} \to 0 \]

\* To satisfy incompressibility \( u = \frac{dy}{dy}, \quad v = -\frac{dx}{dx} \)

\* Look for a similarity solution with inner variable \( \eta = \frac{y}{\sqrt{\text{Re}}} \), \( u = u(\eta) \)

and try to find an ODE \( g(x) \) is to be determined and we expect \( g(x) \to 0 \) as \( \nu \to 0 \)

\( g(x) \) represents the thickness of the boundary changing with \( x \)

\[ u = \frac{dy}{dy} \Rightarrow \psi(x, y) = \int u \, dy \]

\[ = \int_0^1 u(\eta') g(x) \, d\eta' \]

\[ = u_0 g(x) \int_0^1 \frac{u(\eta')}{u_0} \, d\eta' + k(x) \]

Since the plate is a streamline with zero velocity
we may choose $\psi=0$ at $y=0$ ($\eta=0$)

$\Rightarrow 0 = K(x) = 0$

So $\psi(x,y) = \psi(x,0) = u_0 g(x) F(\eta)$

$F(\eta) = \int_0^\eta \frac{u(\eta') d\eta'}{u_0}$

$F(\eta) = 0$

Now calculate all derivatives we need and plug in

$u = \frac{dy}{dy} = u_0 g(x) F'(\eta) \frac{d\eta}{dy} = u_0 g(x) F'(\eta) \frac{1}{g(x)} = u_0 F'(\eta)$

$v = -\frac{dy}{dx} = -u_0 \left[ g'(x) F(\eta) + g(x) F'(\eta) \frac{d\eta}{dx} \right]$

$= -u_0 \left[ g' F + g F' \left( -\frac{g'}{g^2} \right) y \right]$

$= u_0 g' \left[ -F + F' \frac{y}{g} \right] = u_0 g' \left[ -F + F' \frac{g'^2}{g^2} \right]$

$\frac{du}{dx} = u_0 F''(\eta) \frac{d\eta}{dx} = u_0 F'' \left( -\frac{g'}{g^2} \right) y$

$= -u_0 F'' \left( \frac{g'}{g^2} \right) \eta$
\[ \frac{\partial u}{\partial y} = u_0 F'' \quad \frac{\partial^2 u}{\partial y^2} = u_0 F'' \]

\[ \frac{\partial^2 u}{\partial y^2} = u_0 \frac{F'''}{g} \quad \frac{\partial^2 u}{\partial y^2} = u_0 \frac{F'''}{g^2} \]

\[ \text{Ax-momentum:} \]

\[ u_0 F' \left( -u_0 \frac{F''}{g} \right) + u_0 g \left( -F + F \frac{F''}{g} \right) \frac{u_0 F''}{g} \]

\[ \Rightarrow u_0 \frac{F'''}{g^2} \]

\[ -u_0^2 \frac{F'}{g} \frac{F'}{g} + u_0^2 \frac{F''}{g} + u_0^2 \frac{F''}{g'q_1} \]

\[ \Rightarrow u_0 \frac{F'''}{g^2} \]

Multiply by \( \frac{g^2}{\partial u_0} \) \( \Rightarrow \)

\[ F'''' \sim \frac{u_0}{g} g q_1 F' \frac{F''}{g} \quad F(0) = 0 \]
For this to be an ODE in $y \Rightarrow$

\[
\frac{U_0}{V} gg' = \text{constant} = k \quad (\text{choose } k = 1)
\]

\[
qq' = \frac{x^2}{U_0} \Rightarrow \frac{1}{2} q^2 = \frac{U_0}{U_0} x + d = \frac{U_0}{U_0} (x + \frac{du}{dV})
\]

Now recall that derivatives of $u$ are

\[
\frac{du}{dy} = u_0 f' \quad \frac{du}{dx} = -u_0 f'' q' q \quad \frac{d^2 u}{dy^2} = u_0 f''' q^2
\]

and it does not make sense that these would go to infinity except at the sharp leading edge when $x = 0$.

\[
\Rightarrow \text{choose } d = 0 \quad g(x) = \left(\frac{2u x}{U_0}\right)^{1/2}
\]

Thickness of fluid changing with $x$ goes to zero with $x$ as $x^{1/2}$. 
We arrive at the ODE

$$F'''(\eta) = -F(\eta)F''(\eta)$$

$$F(\eta) = \int_0^\eta \frac{u(\eta')}{u_0} d\eta'$$

$$F(0) = 0 \quad \text{(plate is a streamline, } v = 0, \text{ no slip)}$$

$$F'(\eta) \bigg|_{\eta = 0} = \frac{u(\eta)}{u_0} \bigg|_{\eta = 0} = 0 \quad \text{(no slip)}$$

$$F'(\eta) \bigg|_{\eta = \infty} = \frac{u(\eta)}{u_0} \bigg|_{\eta = \infty} = 1 \quad \text{(} u = u_0 \text{)}$$

Numerical Solution required. Figure 8.8

Matching condition!
The thickness of the boundary layer:

\$ \Sigma \text{like } \frac{y}{\delta_{EP}} = O(1) \text{ in ODE example} \$

\[ \Rightarrow y = O(\epsilon P) \]

\[ \frac{y}{\delta(x)} \leq \frac{y}{\delta_0} \left( \frac{2\epsilon \gamma x}{u_0} \right)^{1/2} = O(1) \]

\[ \Rightarrow y = O\left( \left( \frac{\gamma x}{u_0} \right)^{1/2} \right) = \delta \]

We can also calculate the Drag on the plate:

\[ \frac{D}{b} = \frac{2}{L} \int_0^L \frac{t_{xy}}{y=0} \, dx \]

For a long plate plate of length \( L \):

\[ \text{2 factor of 2 for both sides} \]

\[ t_{xy} \bigg|_{y=0} = \mu \left( \frac{\partial u}{\partial y} + \frac{1}{v} \frac{\partial v}{\partial x} \right) \bigg|_{y=0} = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} \]
\[ v = \frac{dy}{dx} = u_0 g \left[ -f + f' \right] \]

\[ \frac{dv}{dx} = \text{after cancellations} \]

\[ = u_0 g'' I F' + u_0 g' n F'' \frac{dn}{dx} - u_0 g'' F \]

\[ \frac{dv}{dx} \bigg|_{y=0} = 0 \quad \text{since} \quad F'(0) = 0, \quad F(0) = 0 \]

\[ \eta = 0 \quad \text{when} \quad y = 0 \]

\[ \frac{mdu}{dy} \bigg|_{y=0} = 0 \]

\[ = m u_0 f''(\eta) \frac{1}{g(x)} \bigg|_{y=0} \]

\[ = m u_0 u_0^{1/2} f''(0) \]

\[ \left( 2^\frac{n}{2} x \right)^{1/2} \]

\[ \frac{D}{b} = \frac{2m u_0^{3/2} f''(0) \int_0^1 x^{-1/2} \, dx}{\left( 2^n \right)^{1/2}} \]
\[ D = \frac{2\sqrt{2} \rho \nu^{1/2}}{b} \left[ U_0^{3/2} f''(0) \right]^{1/2} L^{1/2} \]

proportional to \( L^{1/2} \) (not \( L \))

For one side of the plate:

\[ f''(0) \frac{2\sqrt{2}}{z} = 0.4696 \frac{\sqrt{z}}{2} \]

\[ \frac{D}{b} = 0.664 \rho \nu^{1/2} U_0^{3/2} L^{1/2} \left( \text{in eng. books} \right) \]

In terms of \( \text{Re}_L = \frac{U_0 L}{\nu} \)

\[ \frac{D}{b} = \sqrt{2} \rho \nu^{1/2} U_0^{3/2} L^{1/2} f''(0) \frac{U_0^{1/2} L^{1/2}}{U_0^{1/2} L^{1/2}} = \frac{\sqrt{2} \rho U_0^2 L f''(0)}{\text{Re}_L^{1/2}} \]

with Drag coefficient

\[ C_D = \frac{D}{bL} = \frac{1}{\rho U_0^2} \left( \text{is the } \frac{1}{2} \text{ there?} \right) \]
Falkner–Skan Similarity Solution

What inviscid external (outer) flows $U_e(x)$ lead to a similarity solution to the boundary layer equations:

\[
\frac{u du}{dx} + v \frac{du}{dy} = U_e(x) \frac{du}{dx} + x \frac{d^2 u}{dy^2}
\]

\[
\frac{dp}{dy} \approx 0, \quad \frac{du}{dx} + \frac{dv}{dy} = 0
\]

\[
u = \frac{dy}{dx} = U_e(x) F'(\eta), \quad \eta = \frac{y}{U_e(x)}
\]

where $U_e(x)$ and $S(x)$ are undetermined.

\[
\eta = \int^y_{\eta_0} \frac{dy'}{dy'}, \quad dy' = S(x) U_e(x) \int^1_{U_e(x)} \frac{u}{U_e(x)} \frac{d(y')}{S(x)}
\]

\[
= U_e(x) S(x) \int^\eta_{\eta_0} F'(\eta') d\eta'
\]

(not such great notation)

\[
\eta = U_e(x) S(x) \int^\eta_0 F'(s) ds
\]
Homework: Verify the above

What are α, β for the problem we solved; does it check?

Can we find other solutions?