Chapter 3: Equivalent Statements

* $A$ is row equivalent to $I$
* $A^{-1}$ exists
* $\det(A) \neq 0$
* $Ax = b$ has a unique solution for any $b$

Chapter 4: Extension for $A_{m \times n}$ (includes the $n \times n$ case)

* $Ax = b$ has one or more solutions if $b$ is in the "column space" of $A$
* $Ax = b$ has one or more solutions if $b$ is a linear combination of the columns of $A$. 

A
These are the same, and we need to understand

What is a column space? More generally, what is a vector space? (The columns are vectors)

What is a linear combination of vectors?

[Previewing] Important characteristics of a vector space are that it is:

(i) Closed under addition
(ii) Closed under scalar multiplication
(iii) Contains the zero vector

Important concepts pertaining to vectors:

(a) Linear combination
(b) Linear independence
(c) Linear dependence
Two familiar vector spaces are $\mathbb{R}^2$ and $\mathbb{R}^3$.

$\mathbb{R}^2$ is the vector space of all ordered pairs of real numbers, e.g.,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\mathbb{R}^3$ is the vector space of all ordered triplets of real numbers,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$(\mathbb{R}^2)$ and $\mathbb{R}^3$ contain the zero vector: $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Closure under addition: when we add two vectors, we obtain a new vector in $(\mathbb{R}^2)$ or $\mathbb{R}^3$.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

Closure under scalar multiplication: when we multiply a vector in $(\mathbb{R}^2)$ or $\mathbb{R}^3$ by a scalar, we obtain a new vector in $(\mathbb{R}^2)$ or $\mathbb{R}^3$. 

\[ \]
\[ a \mathbf{u} = a \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix} \]

The length of a vector in \( \mathbb{R}^3 \):
\[ |\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} = (\mathbf{u} \cdot \mathbf{u})^{1/2} = (\mathbf{u}^T \mathbf{u})^{1/2} \]
\[
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1^2 + u_2^2 + u_3^2
\]

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**Linear Dependence of Vectors in \( \mathbb{R}^2 \)**

Two vectors \( \mathbf{u}, \mathbf{v} \) in \( \mathbb{R}^2 \) are linearly dependent if
\[ a \mathbf{u} + b \mathbf{v} = \mathbf{0} \quad \text{for} \quad a, b \text{ not both zero} \]

This makes sense since \( a = \frac{-b}{a} \cdot \mathbf{v} \)

\( a, b \) not both zero

means \( \mathbf{u} \) is a scalar multiple of \( \mathbf{v} \), i.e.

they are on the same line
Linear Dependence of Vectors in $\mathbb{R}^3$

Three vectors $u, v, w$ in $\mathbb{R}^3$ are linearly dependent if $au + bv + cw = 0$ for $a, b, c$ not all zero.

Then one is a linear combination of the other two, e.g.,

$$u = -\frac{1}{a}v - \frac{c}{a}w, \quad a \neq 0$$

Linear Independence

$\mathbb{R}^2$: 2 vectors in $\mathbb{R}^2$ are linearly independent if $au + bv = 0 \Rightarrow a = b = 0$

$\mathbb{R}^3$: 3 vectors in $\mathbb{R}^3$ are linearly independent if $au + bv + cw = 0 \Rightarrow a = b = c = 0$
Example: \( u = (3, -1, 2) \) \( v = (5, 4, -6) \) \( w = (2, 3, -4) \)

Are these vectors linearly dependent or independent?

Let's use the definition:

\[ a u + b v + c w = 0 \quad \text{same as} \]

\[
\begin{bmatrix}
u & v & w
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Check it!

or we might write \( Ax = 0 \)

Each column of \( A \) is one of our given vectors, \( x \) is a vector of unknown coefficients.

We need to know: what is the solution for \( x \)?
Since this system is square, then if \( \det(\mathbf{A}) \neq 0 \), then \( \mathbf{c} = 0 \) is the unique solution. \( \Rightarrow \) linear independence of the columns of \( \mathbf{A} \) (the given vectors)

but if \( \det(\mathbf{A}) = 0 \), then there are infinitely many solutions to \( \mathbf{A}\mathbf{c} = 0 \) \( \Rightarrow \) linear dependence of the columns of \( \mathbf{A} \)

\[
\det(\mathbf{A}) = \begin{vmatrix} 3 & 5 & 8 \\ -1 & 4 & 3 \\ 2 & -6 & -4 \end{vmatrix}
\]

\[
= 3 \begin{vmatrix} 4 & 3 \\ -6 & -4 \end{vmatrix} - 5 \begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} + 8 \begin{vmatrix} -1 & 4 \\ 2 & -6 \end{vmatrix}
\]

\[
= 0
\]

\( \Rightarrow \) linear dependence
Let us use $GE$ to find $c = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

\[
\begin{bmatrix}
3 & 5 & 8 & 0 \\
-1 & 4 & 3 & 0 \\
2 & -6 & -4 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 4 & 3 & 0 \\
3 & 5 & 7 & 0 \\
2 & -6 & -4 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 4 & 3 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

$0a + 0b + 0c = 0$  
let $c = s$

$\mathbf{b}_2 = -s$, $-a + 4b + 3c = 0$

$-a = -4b - 3c$

$\mathbf{c} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix}$

$4s - 3s = s$
\[ C = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \]

Let's check for \( s = 2 \)

\[ C = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \]

\[ A \cdot C = \begin{bmatrix} 3 & 5 & 8 \\ -1 & 4 & -2 \\ 2 & -6 & -4 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ 3(-2) + 5(-2) + 8 \cdot 2 = 0 \quad \checkmark \]

\[ -1(-2) + 4(-2) + 3 \cdot 2 = 0 \quad \checkmark \]

\[ a(-2) - 6(-2) - 4 \cdot 2 = 0 \quad \checkmark \]
Example \( \mathbf{u} = (0, 3, 2), \mathbf{v} = (1, 5, 4), \mathbf{w} = (-1, -2, 3) \)

Are these vectors linearly dependent or independent?

Use definition: \( a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0} \)

Is \( a = b = c \), or not?

\[ \begin{align*}
\text{independence} & \quad \text{dependence} \\
a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = & \quad \mathbf{0} \\
\begin{bmatrix} u \\ v \\ w \end{bmatrix} 
& \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\end{align*} \]

\[ A \mathbf{c} = \mathbf{0} \quad \text{with} \]

\[ A = 
\begin{bmatrix}
0 & 1 & -1 \\
3 & 5 & -2 \\
2 & 4 & 3
\end{bmatrix}
\]

\[ \mathbf{c} = 
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
\]

Since \( A \) is square, we can check the determinant
\[ \text{det}(A) = 0 \begin{vmatrix} 5 & -2 & 3 \ 4 & 3 & 1 \end{vmatrix} -1 \begin{vmatrix} 3 & -2 & 3 \ 2 & 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} 3 & 5 & 3 \ 2 & 3 & 1 \end{vmatrix} = -1(9+4) - (12-10) = -13 - 2 = -15 \]

So we know \( a = b = c = 0 \) and the given vectors are linearly independent.

Please check with GE!

A basis for \( \mathbb{R}^3 \) is any three linearly independent vectors \( x, y, z \) (or \( u, v, w \)).

Any other vector in \( \mathbb{R}^3 \) can be written as a linear combination of basis vectors.

The most common basis is \( (1,0,0), (0,1,0), (0,0,1) \) but we could just as well use \( (0,3,2), (1,5,4), (-1,-2,3) \).
Let's show that \( p \) can be written as a linear combination of the basis vectors \( u = (0, 3, 2) \), \( v = (1, 5, 4) \), \( w = (-1, -2, 3) \).

\[ a u + b v + c w = p \quad \text{same as} \]
\[ \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad \text{same as} \]
\[ \begin{bmatrix} a \\ b \\ c \end{bmatrix} = p \quad A \quad \text{square} \]

We already showed that \( \det(A) \neq 0 \) \([-15]\).

\[ \Rightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = p \quad \text{has a unique solution} \]

for any \( p \).

Given a specific \( p \), use GE to find the unique coefficients \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \).
Summary

* 3 vectors \( u, v, w \) in \( \mathbb{R}^3 \) are linearly independent if

\[
au + bv + cw = 0 \Rightarrow a = b = c = 0
\]

Then \[
\begin{bmatrix}
  u \\
v \\
w
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

or \( A^T c = 0 \) has a unique soln. \( c = 0 \)

and \( \det(A) \neq 0 \)

It follows that \( u, v, w \) are a basis for \( \mathbb{R}^3 \)

and \( A^T c = p \) has a unique soln \( [\text{stand by 6E}] \)

\( p \) is a linear combination of the columns

of \( A \) : \( u, v, w \)

Notice that this is the same as \( A x = b \)!
One of the statements we started with:

\[ Ax = b \]

has one or more solutions if

\[ b \]

is a linear combination of the columns of \( A \).

In our example, \( A \) is square, \( \text{det}(A) \neq 0 \)
and so there is one unique solution.
3 vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) in \( \mathbb{R}^3 \) are linearly dependent if \( a \mathbf{u} + b \mathbf{v} + c \mathbf{w} = \mathbf{0} \),\( \Rightarrow a, b, c \) not all zero.

Then \[ \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \]

or \( A \mathbf{c} = \mathbf{0} \) has an infinite number of solutions and \( \det (A) = 0 \).

It follows that \( A \mathbf{c} = \mathbf{p} \)

has either no solution or an infinite number of solutions.

\[ \text{an infinite number of } \mathbf{p} \text{ is a linear combination of the columns of } A \]