\[ \varepsilon^2 y'' = \phi(x) y \quad \phi(x) = (x^2 + ax)^{\frac{3}{2}} \quad a > 0 \quad y(\infty) = 0 \]

"Turning point" at \( x = 0, \ x = -a \) where \( \phi(x) = 0 \) and there is no small parameter.

1) Physical Optics WKB for \( \varepsilon \to 0^+ \)

\[ y(x) \sim \exp \left[ \frac{S_0}{\varepsilon} + S_1 \right] \left[ 1 + \varepsilon S_2 + \cdots \right] \]

\[ S_0 = -\int_0^x (t^2 + at)^{\frac{3}{2}} dt, \quad S_1 \sim -\frac{1}{2} \ln(x^2 + ax) \]

Choose sign for \( S_0 \) since \( y(\infty) = 0 \)

The conditions for the asymptotic series and truncation:

\[ S \ll S_0 \quad \varepsilon S_2 \ll S_1 \quad \varepsilon S_2 \ll 1 \]

\[ \implies x \gg \varepsilon^{-\frac{2}{3}} \]

2) \( x \ll 1 \quad \varepsilon \to 0^+ \)

\[ \varepsilon^2 y'' = (x^2 + ax) y \sim ax y \]

with \( t = \varepsilon^{-\frac{2}{3}} a^{-\frac{1}{3}} x \quad \implies \)
\[ \frac{d^2 y}{dt^2} = by \quad \Rightarrow \quad y(t) = D A i(t) + E B i(t) \]
\[ = D A i(\varepsilon^{-\frac{2}{3}} a \frac{1}{3} x) + E B i(\varepsilon^{-\frac{2}{3}} a \frac{1}{3} x) \]
\[ A i(t) = 1 + \sum_{n=1}^{\infty} \frac{t^{3n}}{2 \cdot 3 \cdot \cdots \cdot (3n-1)(3n)} \]
\[ B i(t) = t + \sum_{n=1}^{\infty} \frac{t^{3n+1}}{3 \cdot 4 \cdot \cdots \cdot (3n)(3n+1)} \]

converges for all \( t \).

* Can we match in the middle.

* 3 coefficients, 1 has left

* Matching must determine 2 coefficients.
Let consider first \( \lim_{t \to \infty} y(t) \). What does \( t \to \infty \) mean?

\[
\begin{align*}
\varepsilon^{2/3} a \frac{1}{3} \times & \to \infty \quad \text{if} \quad \varepsilon \frac{x}{x^{2/3}} \to \infty
\end{align*}
\]

requires \( x \gg \varepsilon^{2/3} \) good!

Instead of trying to analyze the series, we can go back to the ODE itself:

\[
\frac{d^2 y}{dt^2} = ty, \quad t \to \infty
\]

What kind of a point is \( t = \infty \)?

Let \( t = \frac{1}{5} \) \( \Rightarrow \)

\[
\left( 5^4 \frac{d^2 y}{d5^2} + 25^3 \frac{dy}{d5} \right) = \frac{1}{5} y
\]
\[ \delta = 0 \text{ is ISP } \Rightarrow t = \infty \text{ in any ISP} \]

So we know what to do:

Let \( y(t) = \exp \left[ \sum S_n(t) + S_1(t) + \ldots \right] \)

where now the small parameter in internal \( \frac{1}{t} \)

and we find the leading behavior:

\[ A(t) \sim -\frac{1}{\sqrt{\pi}} + t^{-\frac{1}{4}} \exp \left[ \frac{-2t^{3/2}}{3} \right], \quad t \to \infty \]

\[ B(t) \sim \frac{1}{\sqrt{\pi}} + t^{-\frac{1}{4}} \exp \left[ \frac{2t^{3/2}}{3} \right], \quad t \to \infty \]

Now the coefficients \( \frac{1}{2\sqrt{\pi}}, \frac{1}{\sqrt{\pi}} \) CANNOT

be determined by ISP analysis.

But it turns out that \( A(t), B(t) \) have integral representatives and the coefficients \( \frac{1}{2\sqrt{\pi}}, \frac{1}{\sqrt{\pi}} \)

are determined by asymptotic analysis of these integrals.

\[ \boxed{BO Ch. 6} \]
Since $f \to 0 \Rightarrow e^{-\frac{2}{3}a \frac{1}{3}x} \to 0 \Rightarrow x \gg e^{\frac{2}{3}}$

we find $\lim_{b \to \infty} y_m(t)$

\[ n \frac{D}{2\sqrt{\pi}} \left( e^{-\frac{2}{3}a \frac{1}{3}x} \right)^{-\frac{1}{4}} \exp \left[ -\frac{2}{3} \left( e^{-\frac{2}{3}a \frac{1}{3}x} \right)^{3/2} \right] \]

\[ + \frac{E}{\sqrt{\pi}} \left( e^{-\frac{2}{3}a \frac{1}{3}x} \right)^{-1/4} \exp \left[ \frac{2}{3} \left( e^{-\frac{2}{3}a \frac{1}{3}x} \right)^{3/2} \right] \]

\[ e^{\frac{2}{3}} \ll x \ll 1 \]

Now we need to find $\lim_{x \to e^{\frac{2}{3}}} y_m(x)$

* Do they look the same? YES
* Is there an overlap region? YES
* Is the overlap region $e^{\frac{2}{3}} \ll x \ll 1$? NO
\[ y_{\text{ext}}(x) = C (x^2 + ax)^{-1/4} \exp \left[ -\frac{1}{\xi} \int_0^x \left( t^2 + at \right)^{1/2} dt \right] \]

\[ \lim_{x \to E^{-2\beta} \to 0^+} y_{\text{ext}}(x) = \lim_{x \to E^{-2\beta} \to 0^+} C (ax)^{-1/4} \left\{ 1 - \frac{x}{4a} + \ldots \right\} \]

\[ x \exp \left[ -\frac{1}{\xi} \int_0^x (at)^{1/2} \left\{ 1 - \frac{t}{2a} + \ldots \right\} dt \right] \]

\[ = \lim_{x \to E^{-2\beta} \to 0^+} C (ax)^{-1/4} \left\{ 1 - \frac{x}{4a} + \ldots \right\} \]

\[ x \exp \left[ -\frac{1}{\xi} \left( \frac{2}{3} a^{1/2} x^{3/2} + \frac{1}{5} \frac{1}{\sqrt{a}} x^{5/2} + \ldots \right) \right] \]

So clearly we can truncate the expansions such that

\[ \lim_{t \to \infty} y_{\text{int}}(t) = \lim_{x \to E^{-2\beta}} y_{\text{ext}}(x) \]

\[ = C (ax)^{-1/4} \exp \left[ -\frac{1}{\xi} \frac{2}{3} a^{1/2} x^{3/2} \right] \]
But what does this function mean?

A piece we neglect

$$\exp \left[ -\frac{1}{\varepsilon} \frac{1}{5\sqrt{\alpha}} x^{5/2} \right] \approx \left\{ 1 - \frac{1}{\varepsilon} \frac{1}{5\sqrt{\alpha}} x^{5/2} + \cdots \right\}$$

when we approximate to \( 1 \) \( \Rightarrow \)

$$\frac{x^{5/2}}{\varepsilon} \ll 1 \Rightarrow x^{5/2} \ll \varepsilon$$

$$\Rightarrow x \ll \varepsilon^{2/5}$$

So the overlap region is

$$\varepsilon^{2/3} \ll x \ll \varepsilon^{2/5}$$

\[ \text{not} \quad \varepsilon^{2/3} \ll x \ll 1 \]
Now we have

\[ \lim_{t \to 0^+} y_{n}(t) = \lim_{x \to \frac{2}{3} \epsilon} y_{n}(x) \]

\[ \frac{D}{2\sqrt{\pi}} \left( e^{-\frac{2}{3} a \frac{1}{3} x} \right)^{1/4} \exp \left[ -\frac{2}{3} \left( e^{-\frac{2}{3} a \frac{1}{3} x} \right)^{3/2} \right] \]

\[ + \frac{E}{\sqrt{\pi}} \left( e^{-\frac{2}{3} a \frac{1}{3} x} \right)^{1/4} \exp \left[ \frac{2k}{3} \left( e^{-\frac{2}{3} a \frac{1}{3} x} \right)^{3/2} \right] \]

\[ = C \left( a \epsilon \right)^{1/4} \exp \left[ -\frac{1}{3} \left( a \epsilon \right)^{3/2} x^{3/2} \right] \]

\[ \epsilon^{2/3} \ll x \ll \epsilon^{2/6} \]

\[ \Rightarrow \begin{bmatrix} E = 0 \\ D = 0 \end{bmatrix} \quad \frac{1}{\sqrt{\pi}} (a \epsilon)^{1/6} C \]

Now only one coefficient left!

One boundary condition left!
So altogether

\[ y \sim C (x^2 + ax)^{-1/4} \exp \left[ -\frac{1}{\varepsilon} \int_{0}^{x} (t^2 + at)^{1/2} \, dt \right] \]

\[ x \gg \varepsilon^{2/3}, \quad \varepsilon \to 0^+ \]

\[ y \sim C \sqrt{\pi} \left( a \varepsilon^{-1/6} \right) \text{Ai} \left( \varepsilon^{-2/3} a^{1/3} x \right) \]

\[ 0 \leq x \leq 1, \quad \varepsilon \to 0^+ \]

and there is an overlap region

\[ \varepsilon^{2/3} \ll x \ll \varepsilon^{2/5} \]

To put this together into one formula is non-trivial. Lange's formula in 80

\[ y = y_{\text{in}} + y_{\text{out}} - y_{\text{match}} \]

\[ y_{\text{match}} = C (x)^{-1/4} \exp \left[ -\frac{1}{2} \frac{2}{3} a^{1/2} x^{3/2} \right] \]
have specified the value of

formulas in (10.4.13)
d approximation to

\[
\left( \frac{3}{2e} \alpha \right)^{1/6} = e^{i \pi/4} \left[ \frac{3}{2e} \int_x^0 \sqrt{-Q(t)} \, dt \right]^{1/6}.
\]

Thus, (10.4.15) reduces exactly to the third formula of (10.4.13).

Example 1: Numerical comparison between exact and one-turning-point WKB solutions. In Figs. 10.10 to 10.13 we compare the exact and uniform one-turning-point solutions in (10.4.15) to

\[
e^{-\epsilon^3 y''(x)} = \sinh \left( x \cosh x \right)^3 y(x) \quad [y(0) = 1, y(\infty) = 0]
\]

for \( \epsilon = 0.2, 0.3, 0.5, \) and 1. Note that for this choice of \( Q(x), \) \( a = 1 \) and \( \int_0^x \sqrt{Q(t)} \, dt = 1/2 \sinh x \sqrt{2}. \) The agreement between the exact and the approximate solution is extremely impressive, even when \( \epsilon \) is not small.

Directional Character of the Connection Formula

There is a subtle feature of the solution (10.4.13) to the one-turning-point problem. You will recall that in our analysis of this problem we started with the

\[
\epsilon \to 0^+.
\]

4.13). It should be recalled that the

defined by using the

\[
\epsilon \to 0^+.
\]

Figure 10.10: A comparison of the exact solution to \( \epsilon^{-3} y''(x) = \sinh \left( x \cosh x \right)^3 y(x) \quad [y(0) = 1, y(+\infty) = 0], \) with the approximate solution from a one-turning-point WKB analysis. The WKB approximate formulas are given in (10.4.14) and (10.4.15).
A single particle in a potential well $V(x)$

\[ E^2 y'' = \phi(x) y \quad \phi(x) = V(x) - E \quad E \geq V(x) \]
The energy is quantized if there are discrete values of the eigenvalue \( \lambda = \frac{1}{\epsilon^2} \) for this RSZ problem.

The "connection" is via Airy solutions where \( \phi(x) \) is close to zero. For many "generic" problems, \( \phi(x) \) looks locally linear.

Some of the wave/particle is reflected but a small amount may also be transmitted through the potential barrier.

More details in BÖ Seehra’s 10.5-10.6.

Be careful again about the definition of \( Q(x), V(x) \)!