Math 322 Lecture 16

Topics

1. Complex Fourier Series (Finish Ch. 3)

2. Derivation of the Wave Eqn. and appropriate boundary conditions (Ch. 4)

Complex Fourier Series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$$

with $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \& \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Re-arrange:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{\frac{in\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-\frac{in\pi x}{L}}$$

Change $m = -n$ in the First sum

$$f(x) \sim a_0 + \sum_{m=-1}^{-\infty} \left(\frac{a_{-m}}{2} + \frac{b_{-m}}{2i} \right) e^{-im\pi x/L}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-in\frac{\pi x}{L}}$$

$$a_{-m} = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{-m\pi x}{L} \right) dx$$

$$= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} \, dx \quad = \quad a_m$$

$$b_{-m} = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{-m\pi x}{L} \right) dx$$

$$= \frac{-1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L} \right) dx \quad = \quad -b_m$$

$$f(x) \sim a_0 + \sum_{m=-1}^{-\infty} \left(\frac{a_m}{2} - \frac{b_m}{2i} \right) e^{-im\pi x/L}$$

$$+ \sum_{m=1}^{\infty} \left(\frac{a_m}{2} - \frac{b_m}{2i} \right) e^{-im\pi x/L}$$

So $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$

with $c_0 = a_0$

$$c_n = \frac{a_n}{2} - \frac{b_n}{2i} = \frac{1}{2}\left(a_n + i b_n \right)$$

and by this relation, substituting for a_n, b_n one finds (plug in)

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{+in\pi x/L}\, dx \qquad \text{all } n$$

The same formula for c_n may be found from orthogonality

$$\int_{-L}^{L} e^{-in\pi x/L} e^{im\pi x/L}\, dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

check: $\int_{-L}^{L} dx\, e^{\frac{im\pi x}{L}} f(x) = \int_{-L}^{L} \sum_{n=-\infty}^{\infty} c_n e^{\frac{-in\pi x}{L}} e^{im\pi x/L}\, dx$

$$\int_{-L}^{L} e^{\frac{im\pi x}{L}} f(x)\,dx = \sum_{n=-\infty}^{\infty} \int_{-L}^{L} C_n e^{-\frac{in\pi x}{L}} e^{\frac{im\pi x}{L}}\,dx$$

$$= C_m \cdot 2L$$

Complex Orthogonality in general

$$\int_{-L}^{L} \phi_n(x)\, \phi_m^*(x)\,dx = \begin{cases} 0 & n \neq m \\ I & n = m \end{cases}$$

$*$ means complex conjugate

— book uses bar

$\boxed{\text{Summary of Ch. 3 = Need to know}}$

① $f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

we need $f(x)$ "piecewise smooth"

② How to find a_0, a_n, b_n using orthogonality in $[-L, L]$

③ to replace "$\sim$" by "$=$", need $f(x)$ continuous in $[-L, L]$ and $f(-L) = f(L)$

④ to differentiate term by term, need $f'(x)$ continuous in $[-L, L]$ and $f(-L) = f(L)$

⑤ Integration is always ok term by term with "$=$" sign

⑥ Complex Form of Fourier Series

Now make your own list for

(a) $f(x)$ odd

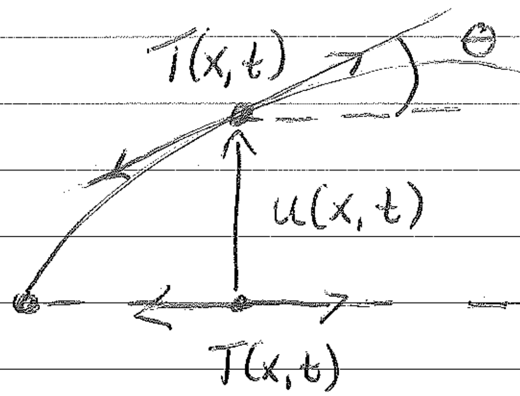
(b) $f(x)$ even

why can we work in $[0, L]$?

(c) $f(x)$ given in $[0, L]$ only

what do we do ...

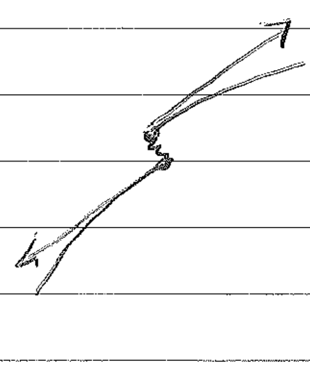
The 1D Wave Equation



Think of a string under tension, e.g. the string of a musical instrument

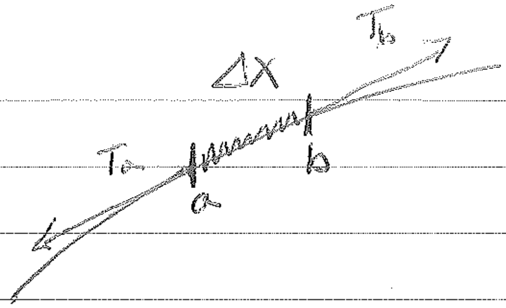
Now make a vertical displacement from the equilibrium position (points on string only move up/down)

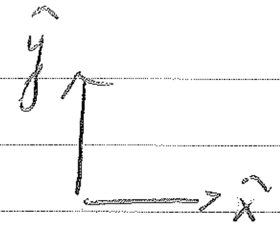
$T(x,t)$ is the <u>magnitude</u> of the tension; the tension is a force that is defined to be tensile.



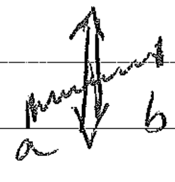
there are opposite forces pulling from either side, equal in magnitude

$u(x,t)$ is displacement from equilibrium in the <u>vertical</u> direction





Conservation of Momentum in the $\hat{y}$-direction



segment of the curve moving up and down

"Control line" analysis

$$\frac{d^2}{dt^2}\int_a^b \rho(x)\,u(x,t)\,dx = T(b)\sin\theta_b - T(a)\sin\theta_a$$

each segment of the string is like a particle

$\rho(x)$ mass per unit length

$\{$ m a = sum of forces $\}$

$$\frac{d^2}{dt^2}\int_a^b \rho(x)\,u(x,t)\,dx = \int_a^b \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x)\right]dx$$

$$\int_a^b \left\{ \frac{\partial^2}{\partial t^2}\left[\rho(x)\,u(x,t)\right] - \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x)\right] \right\} dx = 0$$

$\forall$ a, b $\{$for all a, b$\}$ $\Rightarrow$

$$\rho(x)\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x)\right]$$