The 1D Wave Equation

Think of a string under tension, e.g., the string of a musical instrument.

Now make a vertical displacement from the equilibrium position (points only move up/down on string).

\( T(x,t) \) is the magnitude of the tension.

The tension is a force that is defined to be tensile.

There are opposite forces pulling from either side, equal in magnitude.

\( u(x,t) \) is displacement from equilibrium in the vertical direction.
Conservation of Momentum in the \( \hat{y} \)-direction

"Control line" analysis

\[
\frac{d^2}{dt^2} \int_a^b \rho(x) u(x,t) \, dx = T(b) \sin \theta_b - T(a) \sin \theta_a
\]

each segment of the string is like a particle

\( \rho(x) \) mass per unit length

\[
\sum \text{forces} = ma
\]

\[
\frac{d^2}{dt^2} \int_a^b \rho(x) u(x,t) \, dx = \int_a^b \frac{d}{dx} \left[ T(x,t) \sin \theta(x) \right] \, dx
\]

\[
\int_a^b \left\{ \frac{d^2}{dt^2} \left[ \rho(x) u(x,t) \right] - \frac{d}{dx} \left[ T(x,t) \sin \theta(x) \right] \right\} \, dx = 0
\]

For all \( a, b \)

\[
\rho(x) \frac{d^2}{dt^2} u(x,t) = \frac{d}{dx} \left[ T(x,t) \sin \theta(x) \right]
\]
but \( \hat{t} \) be the tangent vector

\[
\frac{d}{dx} \left( T(x,t) \sin \theta \right) = \lim_{\Delta x \to 0} \frac{\left[ T^+ \hat{t}^+ - T^- \hat{t}^- \right] \cdot \hat{y}}{\Delta x}
\]

\[
T^+ = T(x + \Delta x, t) \quad \hat{t}^+ = \hat{t}(x + \Delta x)
\]

\[
T^- = T(x, t) \quad \hat{t}^- = \hat{t}(x)
\]

\[
\hat{t} = \frac{\cos \theta \hat{x} + \sin \theta \hat{y}}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = \frac{\hat{x} + \frac{du}{dx} \hat{y}}{\sqrt{1^2 + \left( \frac{du}{dx} \right)^2}}
\]

\[
\hat{t} \cdot \hat{y} = \frac{\frac{du}{dx}}{\sqrt{1 + \left( \frac{du}{dx} \right)^2}} = \sin \theta
\]

\[\text{Approximations}\]

1. \( T(x,t) = T_0 \) consistent with

2. \( \hat{t} \cdot \hat{y} \approx \frac{du}{dx} \) small vertical displacements / small angles

\[\left( \frac{du}{dx} \right)^2 \ll 1\]
So we arrive at

\[ \rho(x) \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left( T_0 \frac{\partial u}{\partial x} \right) \]

\[ \frac{\partial^2 u(x,t)}{\partial t^2} \simeq \frac{T_0}{\rho(x)} \frac{\partial^2}{\partial x^2} u(x,t) \]

with \( T_0 / \rho(x) > 0 \implies \)

\[ \frac{\partial^2 u(x,t)}{\partial t^2} \simeq c^2 \frac{\partial^2}{\partial x^2} u(x,t) \]

\( T_0 > 0 \) means the string is always under tension.

Now what are sensible boundary conditions?

\[ \rho(x) = \rho_0 \text{ constant } \Rightarrow c^2 = \frac{T_0}{\rho_0} \]

constant
1. Prescribed values of the displacement at the ends

\[ u(0,t) = f(t) \quad u(L,t) = g(t) \]

2. Vanishing vertical component of the tensile force at the ends

\[ T_0 \frac{du}{dx} \bigg|_{x=0} = 0 \quad T_0 \frac{du}{dx} \bigg|_{x=L} = 0 \]

3. A restoring force at the ends \( \Rightarrow \) the analogy of Newton's law of cooling
$y_s(t)$ is the vertical position of the support (can be moving in time in some prescribed way).

Start with what we understand:

\[ L = \text{equilibrium spring length} \]

\[ (0 = mg - kL) \hat{y} \]

$k$ is the spring constant

\[ F = -kL \hat{y} \] is the restoring force.
In motion: \[ m \frac{d^2 y(t)}{dt^2} = mg - k \left[ y(t) - L \right] \]

\( y(t) \) is measured from the support, assuming that the support does not move.

Now flip upside-down "allow the support to move" add the force due to the string (rope)

\( y(t), y_s(t), \) measured from a fixed reference height

\[ m \frac{d^2 y(t)}{dt^2} = -mg - k \left[ y(t) - (y_s(t) + L) \right] \]

\[ + \frac{To}{dx} \bigg|_{x=0} \]

\[ \frac{du}{dx} > 0 \text{ in picture} \]

extra vertical Force up

\( y - y_s > L \Rightarrow \) spring is stretched

\( y - y_s < L \Rightarrow \) spring is compressed
Now "go massless"

\[ \frac{d}{dx} \frac{du}{dx} \bigg|_{x=0} = \frac{d}{dx} \frac{u}{x=0} - u_E(t) \]

\[ u_E(t) = \text{u_{END}}(t) \text{ prescribed} \]

At the other end \( x=L \)

\[ \frac{d^2}{dt^2} y(t) = -mg - k \left[ y(t) - (y_s(t)+L) \right] \]

\[ + T_0 \left( -\frac{du}{dx} \right) \bigg|_{x=L} \]

Extra force up

\[ \sum \frac{du}{dx} < 0 \text{ \{ in picture} \]

Now "go massless"

\[ -T_0 \frac{du}{dx} \bigg|_{x=L} = k \left[ u_{x=L} - u_E(t) \right] \]