[Math 322 Lecture 18]

Vibrating String with Fixed Ends

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad u(L,t) = 0$$

$0 < x < L, \quad t > 0$

$u(x,0) = f(x)$ initial displacement

$\frac{\partial u}{\partial t}(x,0) = g(x)$ initial velocity

Linear, homogeneous PDE & linear, homogeneous boundary conditions $\Rightarrow$ Separation of Variables is a reasonable strategy

$u(x,t) = \phi(x) h(t)$

Plug in $\Rightarrow$ $\phi(x) \frac{d^2 h(t)}{dt^2} = c^2 h(t) \frac{d^2\phi(x)}{dx^2}$

Translate boundary conditions $\Rightarrow$

$\phi(0) = \phi(L) = 0$

$\rightarrow \quad \frac{1}{c^2 h(t)} \frac{d^2 h(t)}{dt^2} = \frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2} = -\lambda$

$\phi(0) = \phi(L) = 0$

We've solved the x-dependent problem before $\Rightarrow$

$\lambda > 0$

$\lambda = \frac{n^2\pi^2}{L^2}$ $\quad \phi_n(x) = b_n \sin\frac{n\pi x}{L}$ $\quad n = 1, 2, 3 \ldots$

Then for each n $\quad \frac{d^2 h_n(t)}{dt^2} = -\lambda_n c^2 h_n(t)$

$$h_n(t) = \alpha_n \cos\frac{cn\pi t}{L} + \beta_n \sin\frac{cn\pi}{L} t$$

$$u_n(x,t) = \phi_n(x)\, h_n(t)$$

$$= A_n \sin\frac{n\pi x}{L} \cos\frac{cn\pi t}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{cn\pi t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \cos\frac{cn\pi t}{L} + B_n \sin\frac{n\pi x}{L} \sin\frac{cn\pi t}{L}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

$$g(x) = \frac{du}{dt}(x,0) = \sum_{n=1}^{\infty} B_n \left(\frac{cn\pi}{L}\right) \sin\frac{n\pi x}{L}$$

Orthogonality $\Rightarrow$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{L}{cn\pi}\,\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Formal Soln's What does it mean?

Visualize "standing waves" (fixed n, vary t)

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{nc\pi t}{L} + B_n \sin\frac{nc\pi t}{L}\right)$$

amplitude varying in time ; periodic in time ; period $2L/c$

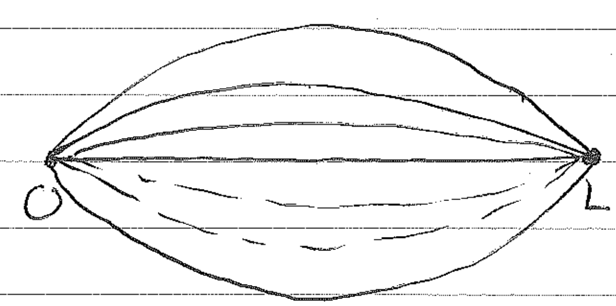
$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n^2 + B_n^2\right)^{1/2} \sin\left(\frac{n\pi c}{L}t + \theta\right)$$

$$\theta = \tan^{-1}\left(A_n/B_n\right)$$

Plot $\sin\frac{n\pi x}{L} \sin\left(\frac{n\pi c t}{L} + \theta\right)$ for fixed n ;

varying in time

$n=1$

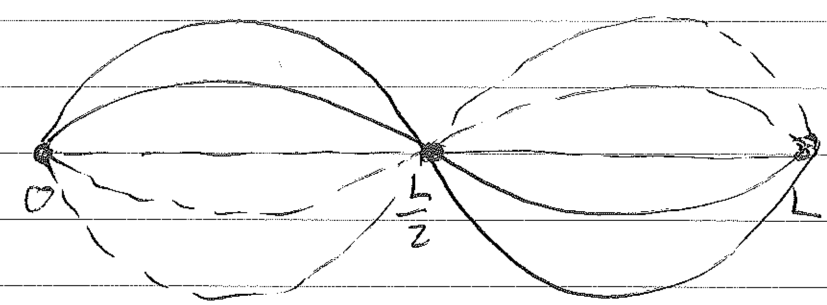


max amplitude 1

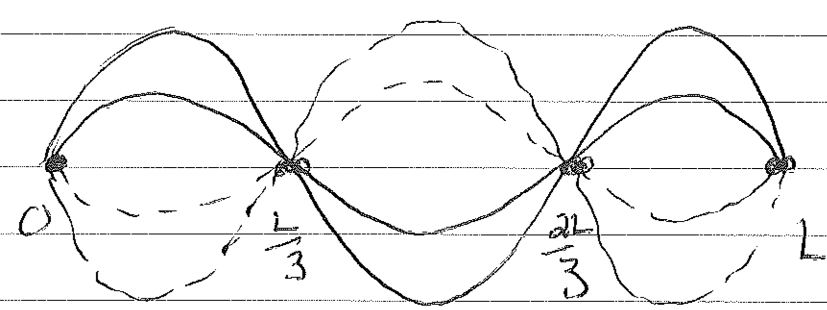
min amplitude -1

2 nodes

$n=2$ (3 nodes)



$n=3$ (4 nodes)



Appendix

Let $f(t) = A_n \cos\omega t + B_n \sin\omega t$ $\quad \omega = nc\pi/L$

How do we write $f(t) = R\sin(\omega t + \theta)$?

Let $A_n = R\sin\theta$ $\quad B_n = R\cos\theta$

$f(t) = R\sin\theta\cos\omega t + R\cos\theta\sin\omega t$

$= R\sin(\omega t + \theta)$ (identity)

$A_n^2 + B_n^2 = R^2\sin^2\theta + R^2\cos^2\theta = R^2$

$A_n / B_n = \tan\theta$

Extensions to higher dimension, e.g. a vibrating membrane (2D):

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right)$$