Review 2nd-order linear homogeneous ODE

(in standard form): \( y''(x) + p(x)y'(x) + q(x)y(x) = 0 \)

with initial conditions: \( y(x_0) = y_0, \ y'(x_0) = y'_0 \)

For the ODE by itself, the solution space is a linear function space of dimension 2.

* Contains zero
* Closed under addition
* Closed under scalar multiplication

* Then the goal is to find 2 linearly independent solutions \( y_1(x), y_2(x) \) and the general solution \( y(x) = c_1 y_1(x) + c_2 y_2(x) \) describes all possible solutions.

* Apply the initial conditions to find \( c_1, c_2 \)
Today] Constant coefficients:

\[ ay'' + by' + cy = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \]

(not standard form)

Constant coefficients \(\Rightarrow\) exponential solutions

\[ y(x) = Ae^{rx} \quad \text{(why?)} \]

Take this as a "trial solution". Plug in

\[ aAe^{rx} + bAe^{rx} + cAe^{rx} = 0 \]

\[ A(\alpha r^2 + \beta r + \gamma) e^{rx} = 0 \]

\[ A \neq 0, \quad e^{rx} \neq 0 \Rightarrow \boxed{\alpha r^2 + \beta r + \gamma = 0} \]

the "characteristic equation."

Solve by quadratic formula with 3 cases:

(i) 2 real different roots

(ii) 1 real repeated root

(iii) 2 complex conjugate roots
Case 1: 2 different real roots \( r_1, r_2 \)

\[ y(x) = C_1 e^{r_1x} + C_2 e^{r_2x} \]

Are they linearly independent?

Check \( W = \begin{vmatrix} e^{r_1x} & e^{r_2x} \\ r_1 e^{r_1x} & r_2 e^{r_2x} \end{vmatrix} = (r_2-r_1) e^{(r_1+r_2)x} \)

\( e^{(r_1+r_2)x} \neq 0 \) for any \( x \)

\( r_2-r_1 \neq 0 \) by assumption (2 different real roots)

\( \Rightarrow \) linear independence

Given any initial conditions \( y(x_0)=y_0, y'(x_0)=y'_0 \),

we can always solve

\[
\begin{bmatrix}
e^{r_1 x_0} & e^{r_2 x_0} \\
e^{r_1 x_0} & r_2 e^{r_2 x_0}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} =
\begin{bmatrix}
y_0 \\
y'_0
\end{bmatrix}
\quad \text{for the unique}
\quad C_1, C_2
\]
Example: \[\frac{d^2y}{dx^2} - 16\frac{dy}{dx} + 30y = 0 \quad y(1) = 1 \quad y'(1) = 3\]

Trial Solution: \[y(x) = Ae^{4x}\]

Characteristic eqn. \[2r^2 - 16r + 30 = 0\]

\[r = \frac{16 \pm \sqrt{16^2 - 4 \cdot 2 \cdot 30}}{4} = 4 \pm 1\]

\[y(x) = C_1 e^{3x} + C_2 e^{5x} \quad -\infty < x < \infty\]

Find \(C_1, C_2\) using initial conditions.

Solve:
\[
\begin{bmatrix}
    e^3 & e^5 \\
    3e^3 & 5e^5
\end{bmatrix}
\begin{bmatrix}
    C_1 \\
    C_2
\end{bmatrix}
= 
\begin{bmatrix}
    1 \\
    3
\end{bmatrix}
\]

\[\begin{bmatrix} 0 & 0 \end{bmatrix}\]
Most famous & classic application is a mass on a spring

![Diagram of a mass on a spring]

Rest length L

Experimental fact: restoring force = \(-kx\) (Hooke's law)

In equilibrium: \(F = ma = 0 = (mg - kL)x\)

where \(x\) is down.

Displace by a distance \(x\) downwards:

\[mg - k(L + x) - \gamma dx(t) = m \frac{d^2x(t)}{dt^2}\]

\[\Rightarrow m \frac{d^2x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + kx(t) = 0\]

with \(x(t_0) = x_0\) (initial displacement)

\[\frac{dx(t)}{dt} \bigg|_{t=t_0} = x_0'\] (initial velocity)

What is \(γ\)? Dimensions of \(k, γ\)?
Case (ii) Repeated Roots

\[ ay'' + by' + cy = 0 \]

Try \[ y = A e^{rx} \]

Plug in \[ \Rightarrow A e^{rx} (ar^2 + br + c) = 0 \]

\[ \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} = r \]

with \[ \sqrt{b^2 - 4ac} = 0. \]

Then \[ y(x) = C_1 e^{rx} + ? \]

There is a method — Reduction of Order — which can be used to find a 2nd linearly independent solution.

Reduction of Order

- works for any linear, homogeneous ODE (even non-constant coefficients & even higher-order)

- must know one solution

- why does it work?
Consider the 2nd-order linear, homogeneous ODE with non-constant coefficients

\[ y''(x) + p(x)y'(x) + q(x)y(x) = 0 \]

[In our particular case \( p(x) = \frac{b}{a} \) and \( q(x) = \frac{c}{a} \)]

Assume we know one solution \( y_1(x) \)

The method: let \( y(x) = v(x)y_1(x) \). For unknown \( v(x) \) and plug in to find \( v(x) \)
[we'll see why it works later]

\[ y(x) = v(x)y_1(x) \quad \Rightarrow \quad y' = v'y_1 + vy_1' \quad \text{by product rule} \]
\[ y'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + av'y_1' + vy_1'' \]

Plug in:

\( v''y_1 + av'y_1' + vy_1'' \)

\[ + p(v'y_1 + vy_1') + qvy_1 = 0 \]

Re-arrange:

\[ v[y'' + py' + qy] + v'y_1 + av'y_1' + pv'y_1 = 0 \]

by assumption because \( y_1 \) satisfies

\[ y_1'' + py_1' + qy_1 = 0 \]
leaves $y, v'' + \left[ a y' + p y \right] v' = 0$

where $y, y', p$ known, $v$ unknown

This is a 1st-order ODE for $v'(x)$ in disguise!

Let $w = v'$, $w' = v''$, then

$$y w' + \left[ a y' + p y \right] w = 0$$

in standard form:

$$w' + \left[ a y' + p y \right] w = 0$$

$$w' + m(x) w = 0, \quad m(x) \text{ known},$$

a 1st-order, linear, homogeneous eqn. For $w = v'$, we can always solve as separable or using the integrating factor method.

For our particular case: $y_1 = e^{ri x}$, $y_2 = r_1 e^{ri x}$, $p = \frac{1}{r_1 a} \implies$

$$a y' + p y = r_1 2 e^{ri x} + \frac{b}{a} e^{ri x} = a r_1 + b$$
but remember also that \( r_1 = -\frac{b}{2a} \), so

\[
dr_1 + \frac{b}{a} = 0 \quad 1
\]

Then \( w'(x) = 0 \implies w(x) = C \)

Then \( v'(x) = w(x) = C \implies v(x) = Cx + D \)

Then \( y(x) = v(x)y_1(x) \implies y(x) = (Cx + D)e^{r_1x} \)

\[
= De^{r_1x} + Cxe^{r_1x} = Dy_1(x) + Cy_2(x)
\]

\( Dy_1(x) \quad Cy_2(x) \)

and now we need to check if \( y_1(x), y_2(x) \)

are linearly independent.

\[
w(y_1, y_2) = \begin{vmatrix}
  e^{r_1x} & xe^{r_1x} \\
  r_1e^{r_1x} & r_1xe^{r_1x} + e^{r_1x}
\end{vmatrix}
\]

\[
= r_1xe + e - r_1xe = e \neq 0 \quad \text{YES}
\]
So the general solution to
\[ ay'' + by' + cy = 0 \] in the case of a real repeated root \( r = \frac{-b}{a} \)

is \( y(x) = C_1 e^{rx} + C_2 xe^{rx}, \quad -\infty < x < \infty \)

\[
\sinh (w_1 y_1, y_2) = \begin{vmatrix} e^{rx} & xe^{rx} \\ r e^{rx} & r_1 xe^{rx} + e^{rx} \end{vmatrix} \neq 0
\]

Then we can find \( C_1, C_2 \) for any initial conditions \( y(x_0) = y_0, \ y'(x_0) = y'_0 \)

by solving
\[
\begin{bmatrix} e^{rx} & xe^{rx} \\ r e^{rx} & r_1 xe^{rx} + e^{rx} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}
\]

guaranteed a unique solution!
So the general solution to
\[ ay'' + by' + cy = 0 \] in the case of a real repeated root \( \lambda = \frac{-b}{2a} \) is
\[ y(x) = e^{\lambda x} \int [ y(x) = C_1 e^{\frac{-b}{2a} x} + C_2 x e^{\frac{-b}{2a} x} ] \\
-\infty < x < \infty \]

**Example:** \[ 5y'' - 6y' + \frac{9}{5} y = 0, \ y(1) = 0, \ y'(1) = 1 \]
characteristic eqn. \[ 5r^2 - 6r + \frac{9}{5} = 0 \]

\[ \Rightarrow \lambda = \frac{6 \pm \sqrt{36 - 4 \cdot 5 \cdot \frac{9}{5}}}{2} = \frac{3}{5} \]

\[ y(x) = C_1 e^{\frac{3x}{5}} + C_2 x e^{\frac{3x}{5}} \]

Find \( C_1, C_2 \) using initial conditions.
\[ y(1) = 0 \Rightarrow c_1 e^{\frac{3}{5}} + c_2 e^{\frac{7}{5}} = 0 \]

\[ y'(x) = c_1 e^{\frac{3}{5}x} + c_2 e^{\frac{3x}{5}} + \frac{3}{5} c_2 x e^{\frac{3x}{5}} \]

\[ y'(1) = 1 = \frac{3}{5} c_1 e^{\frac{3}{5}} + c_2 \left(1 + \frac{3}{5}\right) e^{\frac{3}{5}} = 1 \]

**System**

\[
\begin{bmatrix}
  e^{\frac{3}{5}} & e^{\frac{3}{5}} \\
  e^{\frac{3}{5}} & e^{\frac{3}{5}} \\
  \frac{3}{5} e^{\frac{3}{5}} & \frac{8}{5} e^{\frac{3}{5}}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} =
\begin{bmatrix}
  c \\
  1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 1 & 0 \\
  \frac{3}{5} & \frac{7}{5} & -e^{-\frac{3}{5}}
\end{bmatrix}
R_2 = \frac{3}{5} R_1
\]

\[
\begin{bmatrix}
  1 & 1 & 0 \\
  0 & 1 & e^{-\frac{3}{5}}
\end{bmatrix}
\begin{bmatrix}
  c_2 = e^{-\frac{3}{5}} \\
  c_1 = -e^{-\frac{3}{5}}
\end{bmatrix}
\]

\[ y = -e^{-\frac{3}{5}(x-1)} + \frac{3}{5}(x-1) e^{-\frac{3}{5}} \]

Check by plugging in 1.