Conservation of Energy (internal) in an arbitrary volume of solid

"Control Volume" analysis

\[
\frac{d}{dt} \int_V e \, dV = \int_A \mathbf{q} \cdot (-\mathbf{n}) \, dA + \oint_{\partial V} q \, d\mathbf{A}
\]

Each term in this eqn. has dimensions of energy/time or \[\frac{\text{ml}^2}{t^3}\]

energy \[E = \left[ \frac{\text{ml}^2}{t^2} \right]\]

density \[\varepsilon = \left[ \frac{m}{lt^2} \right]\]

All quantities are functions of 3 space dimensions and time: \(e(x,t), \mathbf{q}(x,t), Q(x,t)\)

\(q(x,t)\) is the energy flux \(\mathbf{q}\) is a vector \(\mathbf{q} \cdot (-\mathbf{n})\) is a scalar
"Easy" cases to think about

\[ q_t^n \leftarrow q_t \rightarrow \]

\[ q_t^n = 0 \quad \Rightarrow \quad \text{"energy w"} \]

\[ q_t^n (-\hat{n}) < 0 \quad \Rightarrow \quad \text{"energy w"} \]

**Dimensional Analysis**

\[ \int_{A} q_t^n (-\hat{n}) \, dA \text{ must have } \left[ \frac{m \cdot l^2}{t^3} \right] \]

\[ \Rightarrow \left[ q_t^n \right] = \left[ \frac{m}{t^3} \right] \]

\[ \int_{A} Q \, dt \text{ must be } \left[ \frac{m \cdot l^2}{t^3} \right] \]

\[ \Rightarrow \left[ Q \right] = \left[ \frac{m}{l \cdot t^3} \right] \]
1. Internal energy depends on
   (i) density of the solid material (how much material after integration over volume)
   (ii) temperature (a measure of equilibrium)

\[ \rightarrow e(x,t) \propto \rho(x,t) T(x,t) \]

\[ \begin{bmatrix} \frac{m}{lt^2} \\ \frac{m}{lt^3} \end{bmatrix} = \begin{bmatrix} \frac{m}{l} \\ \frac{m}{l^2} \end{bmatrix} \begin{bmatrix} 0 & K \end{bmatrix} \]

hence there must be a proportionality coefficient

\[ c(x,t) \text{ with } [c] = \begin{bmatrix} \frac{l^2}{t^2} \\ \frac{t^2}{0K} \end{bmatrix} \]
Physical meaning of \( c(x, t) \).

Compare

\[
[E] = \left[ \frac{m \cdot l^2}{t^2} \right] \quad \text{to} \quad [c] = \left[ \frac{l^2}{t^2 \cdot \text{K}} \right]
\]

\( c(x, t) \) represents the amount of energy needed to raise the temperature of a unit mass by 1°C.

c is called the "specific heat coefficient"; it is a thermodynamic variable, a function of 2 of \( \rho, p, T \).

\[ c(\rho, T) = c(x, t) \]

So \( c(x, t) = c(x, t) \rho(x, t) T(x, t) \)

2. What is \( \frac{q}{t} \)?

*Farrier's law (an experimental fact)*: energy flows from hot to cold.

\[ \frac{q}{t} \propto -\nabla T \quad \nabla T \text{ is in the direction from cold to hot} \]
\[
\frac{\partial}{\partial t}(x, t) = -k(x, t) \nabla T(x, t)
\]

\[
\begin{bmatrix}
\frac{m}{t^3} & \left[ \frac{me}{t^3} \right] \\
\end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix} k \end{bmatrix} = \left[ \frac{me}{t^3} \right] ^{\text{thermal conductivity}}
\]

Finally we arrive at

\[
\frac{d}{dt} \int \rho T \, dV = \int k \nabla T \cdot \hat{n} \, dA + \int \nabla \cdot \mathbf{q} \, dV
\]

This is now conservation of energy in control volume

form in terms of temperature \( T \)

Next:

* Use Divergence Thm

* Shrink the volume to a point

* Arrive at the PDE

* Simplify the PDE
Divergence Thm:
\[ \int_V \nabla \cdot \mathbf{F} \, dV = \oint_S \mathbf{F} \cdot \mathbf{n} \, dA \]

where \( \mathbf{F} = \mathbf{F}(\mathbf{x}) \) is a vector field in 3D.

\[ \int_a^b \frac{d}{dx} \mathbf{F} \, dx = \mathbf{F}(b) - \mathbf{F}(a) \]

\text{a generalization of the Fundamental Thm of Calculus in 1D}

\[ \frac{d}{dt} \int_V \mathbf{c} \cdot \mathbf{r} \, dV = \int_V \nabla \cdot \left[ \mathbf{c} \nabla \mathbf{I} \right] \, dV + \oint_S \mathbf{c} \, \mathbf{I} \, dA \]

all volume integrals \( \cdot \) all scalar quantities / integrands
\[ \nabla \cdot \mathbf{v} \text{ arises from (i) a conservation law; (ii) a flux law; (iii) divergence term.} \]

New shrink \( \nabla \cdot \mathbf{v} \) to a point:

\[
\frac{\partial (\rho c T)}{\partial t} = \nabla \cdot (k \nabla T) + q
\]

where each term has dimensions \( \left[ \frac{\text{energy}}{\text{volume} \times \text{time}} \right] \)

\( c(x,t) \rho(x,t) T(x,t) k(x,t) q(x,t) \)

Simplify: For a constant density material in a restricted temperature range (no explosions):

\( \rho = \rho_0, \quad c(\rho,T) \approx c_0, \quad k(\rho,T) \approx k_0 \Rightarrow \)

\[
c_0 \rho_0 \frac{dT}{dt} = k_0 \nabla^2 T + q \quad \text{or}
\]

\[
\frac{dT}{dt} = K \nabla^2 T + \tilde{q}
\]

\( K = \frac{k_0}{c_0 \rho_0} \quad \tilde{q} = \frac{q}{c_0 \rho_0} \quad \text{(drop \( c_0 \rho_0 \))} \)