Math 322 Lecture 20

Regular Sturm Liouville Problems

$L[\phi] = -\lambda \sigma \phi \qquad \phi = \phi(x) \quad a < x < b$

$L = \frac{d}{dx}\left[p(x) \frac{d}{dx} \right] + q(x)$

$p(x), q(x), \sigma(x)$ real continuous $a \le x \le b$

$p(x), \sigma(x) > 0 \quad a \le x \le b$

Boundary Conditions

$\beta_1 \phi(a) + \beta_2 \phi'(a) = 0$

$\beta_3 \phi(b) + \beta_4 \phi'(b) = 0$

[1] Symmetry of the Operator + Boundary Conditions

$$\int_a^b \bar{v} L[u] \, dx = \int_a^b u L[\bar{v}] \, dx$$

where $\bar{\ }$ is complex conjugate

Integration by parts { uses continuity of $p(x)$ and boundary conditions }

$\boxed{2}$ λ real

Uses symmetry, $\sigma(x) > 0$

$\boxed{3}$ For each λ, there is only one linearly independent eigenfunction

[not true for periodic boundary conditions!]

Assume $L[\phi_1] = -\lambda_0 \sigma \phi_1$, $L[\phi_2] = -\lambda_0 \sigma \phi_2$

with ϕ_1, ϕ_2 linearly independent for $\lambda = \lambda_0$;

and ϕ_1, ϕ_2 satisfy the boundary conditions

By definition of linear independence on $a \le x \le b$:

$$\left.\begin{aligned} c_1 \phi_1(x) + c_2 \phi_2(x) &= 0 \\ c_1 \phi_1'(x) + c_2 \phi_2'(x) &= 0 \end{aligned}\right\} \quad a \le x \le b$$

can only be satisfied with $c_1 = c_2 = 0$

Need to show a contradiction

can only be satisfied with $C_1 = C_2 = 0$;

$$\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad a \le x \le b$$

has unique soln. $C_1 = C_2 = 0$ only if

$$W(x) = \phi_1(x)\,\phi_2'(x) - \phi_2(x)\,\phi_1'(x) \neq 0$$

We also know that $W(x)$ satisfies the 1^{st}-order ODE

$$\frac{dW(x)}{dx} + \frac{p'(x)}{p(x)} W(x) = 0 \qquad \text{in } a \le x \le b$$

as can be checked by direct substitution

* compute $dW(x)/dx$ (2^{nd} order derivatives of ϕ)
* use $L(\phi_1) = -\lambda_0 \sigma \phi_1$ $\quad L(\phi_2) = -\lambda_0 \sigma \phi_2$

with solution

$$W(x) = C_W \exp\left[- \int \frac{p'(x)}{p(x)}\, dx \right] \qquad a \le x \le b$$

Since the exponential function is never zero $\Rightarrow$

$W(x) = 0 \quad \forall \quad a \le x \le b \qquad \{\text{if } C_W = 0\}$

or

$W(x) \neq 0 \quad$ for any $\quad a \le x \le b \qquad \{\text{if } C_W \neq 0\}$

Using the boundary conditions we can compute

$$W(a) = \phi_1(a)\left(-\frac{\beta_1}{\beta_2}\right)\phi_2(a) - \left(-\beta_1/\beta_2\right)\phi_1(a)\phi_2(a) = 0$$

$$W(b) = \phi_1(b)\left(-\frac{\beta_3}{\beta_4}\right)\phi_2(b) - \left(-\beta_3/\beta_4\right)\phi_1(b)\phi_2(b) = 0$$

and therefore $W(x) = 0$ for all $a \le x \le b$

$\Rightarrow$ $\phi_1(x)$ $\phi_2(x)$ cannot be linearly independent in $a \le x \le b$

$\boxed{4}$ The eigenfunctions are real (may be chosen real)

Assume $\phi = U + iV$ is complex

We know $\lambda = \lambda_0$ is real by $\boxed{2}$

and $L(\phi) = -\lambda_0 \sigma \phi \implies$

$L[U + iV] = -\lambda_0 \sigma (U + iV)$ can only be

satisfied if $L[U] = -\lambda_0 \sigma U$;

$$L[V] = -\lambda_0 \sigma V$$

also U, V must separately satisfy the bundary contins

$$\beta_1 [U(a) + iV(a)] + \beta_2 [U'(a) + iV'(a)] = 0$$

$$\implies \beta_1 U(a) + \beta_2 U'(a) = 0$$

$$\beta_1 V(a) + \beta_2 V'(a) = 0$$

$$\beta_3 [U(b) + iV(b)] + \beta_4 [U'(b) + iV'(b)] = 0$$

$$\implies \beta_3 U(b) + \beta_4 U'(b) = 0$$

$$\beta_3 V(b) + \beta_4 V'(b) = 0$$

It follows from $\boxed{3}$ that $U(x)$, $V(x)$ are linearly dependent:

$U + iV = U + iCU = (1+iC)U$ $\quad$ C constant

and we may choose C pure imaginary so that $u = AU$ with A real

$\boxed{5}$ The eigenfunctions are orthogonal with respect to the weight functions $\sigma(x)$

Let λ_1, λ_2 be different, with eigenfunctions $\phi_1(x)$, $\phi_2(x)$ $\Rightarrow$

$L(\phi_1) = -\lambda_1 \sigma \phi_1 \qquad L(\phi_2) = -\lambda_2 \sigma \phi_2$

By symmetry and reality (eigenvectors & eigenfunctions)

$$(L(\phi_1), \phi_2) - (\phi_1, L(\phi_2)) = 0$$

$$(\lambda_1 \sigma \phi_1, \phi_2) - (\phi_1, \lambda_2 \sigma \phi_2) = 0$$

$$\lambda_1 \int_a^b \sigma \phi_1 \phi_2 \, dx - \lambda_2 \int_a^b \phi_1 \sigma \phi_2 \, dx = 0$$

$$(\lambda_1 - \lambda_2) \int_a^b \sigma \phi_1 \phi_2 \, dx = 0 \Rightarrow \int_a^b \sigma \phi_1 \phi_2 \, dx = 0$$

since $\lambda_1 \neq \lambda_2$ by assumption

With orthogonality, we can write generalized eigenfunction expansions:

For $f(x)$ piecewise smooth

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$$

converges to $\frac{[f(x^+) + f(x^-)]}{2}$ $\quad a < x < b$

with $a_n = \frac{1}{I} \int_a^b f(x)\, \phi_n(x)\, \sigma(x)\, dx$

where $I = \int_a^b \phi_n(x)\, \phi_n(x)\, \sigma(x)\, dx$

$$\int_a^b \sigma(x)\, \phi_m(x) \left\{ f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x) \right\} dx$$

$$\int_a^b \sigma(x)\, \phi_m(x)\, f(x)\, dx = \int_a^b \sum_{n=1}^{\infty} a_n \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx$$

$$= \sum_{n=1}^{\infty} \int_a^b a_n \phi_n(x)\, \phi_m(x)\, \sigma(x)\, dx$$

Integration term by term always ok!

$$\int_a^b \sigma(x)\,\phi_m(x)\,f(x)\,dx = I\,\delta_{nm}\,a_n$$

$$a_m = \frac{1}{I}\int_a^b f(x)\,\phi_m(x)\,\sigma(x)\,dx$$

$$I = \int_a^b \phi_n^2(x)\,\sigma(x)\,dx$$

Why would a Chebychev expansion be more useful than a Fourier expansion?

What is a Chebyshev expansion?

Questions for the future ...