Regular Sturm-Liouville Problem

\[ L[\phi] = -\lambda \phi \quad \phi = \phi(x) \quad a \leq x \leq b \]

\[ L = \frac{d}{dx} \left[ p(x) \frac{d}{dx} \right] + q(x) \]

\( p(x), q(x), \phi(x) \) real continuous \( a \leq x \leq b \)

\( p(x), q(x) > 0 \quad a \leq x \leq b \)

Boundary Conditions

\[ \beta_1 \phi(a) + \beta_2 \phi'(a) = 0 \]

\[ \beta_3 \phi(b) + \beta_4 \phi'(b) = 0 \]

\[ \text{Symmetry of the Operator + Boundary Conditions} \]

\[ \int_a^b \bar{v} L[u] dx = \int_a^b u \overline{L[v]} dx \]

where \( \overline{v} \) is complex conjugate

Integration by parts uses continuity of \( p(x) \) and boundary conditions?
1) $\lambda$ real

Uses symmetry, $\sigma(x) > 0$

3) For each $\lambda$, there is only one linearly independent eigenfunction $\phi$.

But true for periodic boundary conditions!

Assume $L[\phi_1] = -\lambda \sigma \phi_1$, $L[\phi_2] = -\lambda \sigma \phi_2$

with $\phi_1, \phi_2$ linearly independent for $\lambda > 0$.

and $\phi_1, \phi_2$ satisfy the boundary conditions.

By definition of linear independence on $a \leq x \leq b$:

$$
\begin{align*}
\phi_1(x) + \phi_2(x) & = 0 \quad a \leq x \leq b \\
\phi_1'(x) + \phi_2'(x) & = 0 
\end{align*}
$$

Can only be satisfied with $c_1 = c_2 = 0$

Need to show a contradiction.
can only be satisfied with \( \xi_1 = \xi_2 = 0 \). 

\[
\begin{bmatrix}
\phi_1 & \phi_2 \\
\phi_1' & \phi_2'
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\quad a \leq x \leq b
\]

has unique soln, \( \xi_1 = \xi_2 = 0 \) only if

\[
W(x) = \phi_1(x) \phi_2'(x) - \phi_2(x) \phi_1'(x) \neq 0
\]

We also know that \( W(x) \) satisfies the 1st-order ODE

\[
\frac{dW(x)}{dx} + \frac{p'(x)}{p(x)} W(x) = 0 \quad \text{in} \quad a \leq x \leq b
\]

as can be checked by direct substitution.

* compute \( \frac{dW(x)}{dx} \) (2nd order derivatives of \( \phi \))

* use \( L(\phi_1) = -\lambda_1 \sigma \phi_1 \) \( L(\phi_2) = -\lambda_2 \sigma \phi_2 \)
with solution

\[ w(x) = C_{w} \exp \left[ - \int _{a}^{b} \frac{p'(x)}{p(x)} \, dx \right] \quad a \leq x \leq b \]

Since the exponential function is never zero \( \Rightarrow \)

\[ w(x) = 0 \quad \forall \ a \leq x \leq b \quad \exists \ i \notin \{1 \} \quad (w = 0^2) \]

or

\[ w(x) \neq 0 \quad \forall \ a \leq x \leq b \quad \exists \ i \notin \{1 \} \quad (w \neq 0^2) \]

Using the boundary conditions we can compute

\[ w(a) = \phi_{1}(a) \left( -\frac{\beta_{2}}{\beta_{4}} \right) \phi_{2}(a) - \left( -\frac{\beta_{3}}{\beta_{4}} \right) \phi_{1}(a) \phi_{2}(a) = 0 \]

\[ w(b) = \phi_{1}(b) \left( -\frac{\beta_{3}}{\beta_{4}} \right) \phi_{2}(b) - \left( -\frac{\beta_{3}}{\beta_{4}} \right) \phi_{1}(b) \phi_{2}(b) = 0 \]

and therefore \( w(x) = 0 \) for all \( a \leq x \leq b \)

\[ \Rightarrow \phi_{1}(x) \phi_{2}(x) \text{ cannot be linearly independent in } a \leq x \leq b \]
The eigenfunctions are real (may be chosen real).

Assume \( \phi = U + iV \) is complex.

We know \( \lambda = \lambda_0 \) is real by \( \square \).

And \( L(\phi) = -\lambda_0 \sigma \phi \Rightarrow \)

\[
L[U + iV] = -\lambda_0 \sigma (U + iV) \text{ can only be satisfied if } \]

\[
L[U] = -\lambda_0 \sigma U, \quad L[V] = -\lambda_0 \sigma V
\]

Also \( U, V \) must separately satisfy the boundary conditions

\[
\beta_1 [U(a) + iV(a)] + \beta_2 [U'(a) + iV'(a)] = 0
\]

\[
\Rightarrow \beta_1 U(a) + \beta_2 U'(a) = 0
\]

\[
\beta_1 V(a) + \beta_2 V'(a) = 0
\]

\[
\beta_3 [U(b) + iV(b)] + \beta_4 [U'(b) + iV'(b)] = 0
\]

\[
\Rightarrow \beta_3 U(b) + \beta_4 U'(b) = 0
\]

\[
\beta_3 V(b) + \beta_4 V'(b) = 0
\]
It follows from [3] that \( U(x), V(x) \) are linearly dependent:

\[
U + iV = U + iCU = (1 + iC)U \quad C \text{ constant}
\]

and we may choose \( C \) pure imaginary so that \( \partial U = AU \) with \( A \) real.

The eigenfunctions are orthogonal with respect to the weight function \( \sigma(x) \).

Let \( \phi_1, \phi_2 \) be different, with eigenfunctions \( \phi_1(x), \phi_2(x) \Rightarrow \)

\[
L(\phi_1) = -1, \sigma \phi_1 \quad L(\phi_2) = -1, \sigma \phi_2
\]

By symmetry and reality (eigenvectors & eigenfunction)

\[
(L(\phi_1), \phi_2) - (\phi_1, L(\phi_2)) = 0
\]

\[
(\lambda_1 \sigma \phi_1, \phi_2) - (\phi_1, \lambda_2 \sigma \phi_2) = 0
\]

\[
\lambda_1 \int_a^b \sigma \phi_1 \phi_2 \, dx - \lambda_2 \int_a^b \phi_1 \sigma \phi_2 \, dx = 0
\]

\[
(\lambda_1 - \lambda_2) \int_a^b \sigma \phi_1 \phi_2 \, dx = 0 \quad \Rightarrow \quad \int_a^b \sigma \phi_1 \phi_2 \, dx = 0
\]

Since \( \lambda_1 \neq \lambda_2 \) by assumption.
With orthogonality, we can write a generalised eigenfunction expansion as:

For \( F(x) \) piecewise smooth, \( F(x) \approx \sum_{n=1}^{\infty} a_n \phi_n(x) \)

converges to \( \frac{1}{2} \left[ F(x^+) + F(x^-) \right] \quad a < x < b \)

with \( a_n = \frac{1}{I} \int_{a}^{b} F(x) \phi_n(x) \sigma(x) \, dx \)

where \( I = \int_{a}^{b} \phi_n(x) \phi_m(x) \sigma(x) \, dx \)

\( \int_{a}^{b} \phi_n(x) \phi_m(x) \, dx = \int_{a}^{b} \frac{\delta_{nm}}{I} \int_{a}^{b} \frac{\phi_n(x) \phi_m(x) \sigma(x) \, dx}{\int_{a}^{b} \phi_n(x) \phi_n(x) \sigma(x) \, dx} \)

\( \int_{a}^{b} \phi_n(x) \phi_m(x) F(x) \, dx = \sum_{n=1}^{\infty} a_n \phi_n(x) \phi_m(x) \sigma(x) \, dx \)
Integration term by term always ok!

\[ \int_{a}^{b} f(x) \phi_m(x) \phi_n(x) \, dx = I \delta_{mn} a_n \]

\[ a_n = \frac{1}{I} \int_{a}^{b} f(x) \phi_m(x) \phi_n(x) \, dx \]

\[ I = \int_{a}^{b} \phi_n^2(x) \phi(n(x) \, dx \]

Why would a Chebyshev expansion be more useful than a Fourier expansion?

What is a Chebyshev expansion?

Questions for the future...