Math 705  

Kelvin Helmholtz

Base Flow

\[ u = u_0 \hat{x} \quad p = p_2 \quad P_2 = P_0 - P_0 g z \]

\[ u = u_1 \hat{x} \quad p = p_1 \quad P_1 = P_0 - P_1 g z \]

Perturbed Problem

\[ u = \nabla \phi_2 \quad z > \delta \]

\[ u = \nabla \phi_1 \quad z < \delta \]

\[ \phi_1 = \phi_0 + \varepsilon \phi_1' + \ldots \quad \varepsilon \to 0 \]

\[ \phi_2 = \phi^0_2 + \varepsilon \phi_2' + \ldots \]

\[ \delta = \varepsilon \delta' + \ldots \]

Plug in

**0(1) terms are the base flow terms**

+ boundary conditions

**0(\varepsilon) terms give the "linearized" stability equations**
Summary

* Linear, constant coefficient PDEs + boundary conditions allow for perturbation

\[ \tilde{s}(x,t) = \tilde{s} e^{ikx} e^{st} \]

* Laplace's equation for \( \phi_1', \phi_2' \) + boundary condition at \( z \to \pm \infty \) gives

\[ \phi_1' = A_1 e^{ikx} e^{st}, \quad \phi_2' = A_2 e^{-ikx} e^{st} \]

* Material surface condition gives

\[ A_1, A_2 \text{ in terms of } (k, s, \tilde{s}), (\phi_1, \phi_2) \]

* Pressure condition at the interface gives the dispersion relation

\[ s = s(k) \]

\[ = s(k \circ (p_1, p_2, u_1, u_2, g)) \]

* Given spatial dependence \( k \) and amplitude \( \tilde{s} \)

\[ \Rightarrow s(k), \phi_1', \phi_2' \text{ eigenvalues, eigenfunctions} \]
Now we need to look at $\delta = R(s)$ to decide if/when we have instability = exponential growth in time.

Eigenmodes are:

* neutrally stable if $s$ is pure imaginary, $\text{Re}(s) = 0$

* decaying if $\text{Re}(s) < 0$

* growing if $\text{Re}(s) > 0$

In our case we have neutrally stable modes for both $\pm$ sign if $\gamma \leq 0$ inside $\left( \right)^{1/2}$:

$$k^2 p_1 p_2 (U_1 - U_2)^2 - k \log (p_1 - p_2)(p_1 + p_2) \leq 0$$

$$1 \log (p_1 - p_2^2) \geq k^2 p_1 p_2 (U_1 - U_2)^2$$

and we call the mode with $\pm$ sign "marginally stable" (transition eigenvalue from neutrally stable to unstable).

Eigenmodes are wavelike in $x$, $t$ and exponentially decaying in $z$. 
\((\frac{1}{k} - 1)^{\frac{1}{2}} > 0 \) \quad \text{for} \quad \sqrt{\frac{1}{k^2} g \left( p_1^2 - p_2^2 \right)} < k^2 p_1 p_2 (u_1 - u_2)^2 \quad (*)

with one growing / one decaying mode.

Condition (*) is a necessary and sufficient condition for instability.

\Rightarrow \text{ the flow is always unstable if}
\n\begin{align*}
U_1 &\neq U_2 \quad \text{because we can find a } k \\
\text{large enough such that} \\
\frac{g \left( p_1^2 - p_2^2 \right)}{k f p_1 p_2} &< (u_1 - u_2)^2
\end{align*}

\Rightarrow \text{unstable at small scales}

(would be "regularized" by \\
\text{viscosity})
Special Cases

Surface Gravity Waves

\( p_z = 0 \quad u_1 = u_2 = 0 \)

\[ s = \pm i \sqrt{(1k_1 g) \frac{1}{j}} \]

with phase velocity

\[ c = i s = \pm j \sqrt{\frac{1}{k_1} (1k_1)} \]

\[ c = \pm \frac{\sqrt{\frac{1}{k_1}}}{\sqrt{1k_1}} \quad \text{"well-known"} \]

Internal Gravity Waves

\( u_1 = u_2 = 0 \)

\[ s = \pm \sqrt{\frac{1k_1 g (p_1 - p_2)}{2(p_1 + p_2)}} \]

\[ s = \pm \sqrt{\frac{1k_1 g (p_2 - p_1)}{2(p_1 + p_2)}} \]

\( \Rightarrow \) instability only if \( p_2 > p_1 \) \( \Rightarrow \)

heavy fluid above light fluid

The eigenfunctions decay with \( z \) so that

the instability is confined to a layer near the interface.
See this instability when salt water is above fresh water.

Instability due to shear \( \rho_1 = \rho_2, U_1 \neq U_2 \)

\[
S = -\frac{1}{2} i k (U_1 + U_2) \pm \frac{1}{2} |k| (U_1 - U_2)
\]

always unstable with phase velocity

\[
c = \frac{\imath S}{|k|} = \frac{1}{2} |k| (U_1 + U_2) = \frac{1}{2} (U_1 + U_2)
\]

(Will be "regularized" by finite layer)

In general there is a stabilizing effect if \( \rho_1 > \rho_2 \) (heavy fluid beneath light fluid) and a destabilizing effect of shear.
Phase speed

A wave \( A_k e^{ikx} e^{st} \) \( s = s(k) \)

\[
= A_k e^{i k \left( x + \frac{st}{k} \right)} = A_k e^{i k \left( x - \frac{ist}{k} \right)}
\]

If \( s \) is pure imaginary:

\[
= A_k e^{ik(x-ct)} \quad c = \frac{is}{|k|}
\]

is the phase speed

\[
\frac{du}{dt} = c \frac{du}{dx}, \quad u(x,t) = A_k e^{ik(x-ct)}
\]

In more than one dimension

\[
c = \frac{is \cdot k}{|k|} = \frac{is \cdot \hat{k}}{k}
\]
\[ s(k) = -i k \left( \frac{\rho_1 u_1 + \rho_2 u_2}{\rho_1 + \rho_2} \right) \]

\[ \pm \sum \left\{ \frac{k^2 \rho_1 \rho_2 (u_1 - u_2)^2}{(\rho_1 + \rho_2)^2} - \frac{1k l g(\rho_1 - \rho_2)}{\rho_1 + \rho_2} \right\}^{1/2} \]