Math for Rayleigh-Bénard Convection

- Cold
  - Small $\Delta T$, laminar
  - Linear temp. profile
  - No motion

- Hot

- Cold
  - $g$

- Hot

- Cold
  - $g$
  - Convective rolls begin $\Delta T$, laminar, new convective rolls

There is an instability to motion, and laminar boundary layers develop.

Can we analyze the stability? Need to include temperature equation (conservation of energy) and buoyancy compressibility effects.
Conservation of mass, momentum, and energy

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad \text{(mass)} \]

\[ \rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \mathbf{F} \quad \text{(momentum)} \]

* We cheated a little when we wrote directly

\[ \nabla \cdot \mathbf{F} = \mathbf{I} = \left[ \frac{\partial \mathbf{u}}{\partial x} + (\nabla \mathbf{u})^T \right] \]

* We didn't derive conservation of energy

Let's revisit
Energy Equation

1st law of thermodynamics: Fluid at rest

\[ d\hat{u} = S_q - S_W \]
change in internal heat added work done
energy in CV per unit mass by CV
per unit mass mass per unit mass

\( S_q \) and \( S_W \) are path dependent

units \[ \text{[} \frac{\text{g}}{\text{t}^2} \text{]} \]

For the fluid in motion:

\[ d\left( \hat{u} + \frac{\hat{u} \cdot \hat{u}}{2} \right) = S_q - S_W \]

kinetic energy/unit mass

Energy Equation (thermal + kinetic)

\[ \frac{\partial}{\partial t} \int_V \rho \left( \hat{u} + \frac{\hat{u} \cdot \hat{u}}{2} \right) \, dV = \int_A \rho \left( \hat{u} + \frac{\hat{u} \cdot \hat{u}}{2} \right) u \cdot \hat{n} \, dA \]

\[ + \frac{\partial S_q}{\partial t} - \frac{\partial S_W}{\partial t} \]
Work has units of force \& velocity.

Work done on CV by pressure forces:

\[ \int_A p \cdot (-\hat{n}) \cdot u \, dA \]

\[
\text{Pressure} \quad \frac{\delta W}{\delta t} = -\int_A p \cdot \hat{n} \cdot u \, dA
\]

\[
\text{Viscous Forces} \quad \frac{\delta W}{\delta t} = \int_A \tau \cdot \hat{n} \cdot u \, dA
\]

\[
\frac{3}{2} \int_A \tau_{ij} u \cdot n_j \, dA
\]

\[
\text{Gravity} \quad \frac{\delta W}{\delta t} = \int_A \rho g \cdot u \, dA
\]
Heat added to CV per unit time:

$$\frac{\delta Q}{\delta t} = \int_A q \cdot (-\hat{n}) \, dA$$

where \( q \) is the heat flux vector. Use divergence theorem to shrink CV to a point:

$$\frac{\partial}{\partial t} \left( \rho (\hat{u} + \frac{u \cdot u}{2}) \right) + \nabla \cdot \left[ \rho (\hat{u} + \frac{u \cdot u}{2}) \hat{u} \right]$$

$$= -\nabla \cdot (\rho \hat{u}) + \nabla \cdot \left[ \rho \left( \nabla \cdot \hat{u} \right) \right] - \nabla \cdot q + \rho g \cdot \hat{u}$$

$$\nabla \cdot \{ \begin{array}{c} u_i \ t_{ij} \end{array} \}$$

We can use:

$$\frac{\partial}{\partial t} (\rho F) + \nabla (\rho F \cdot \hat{u}) = \nabla \cdot F$$

Then using conservation of mass and vector identities..
\[
\frac{\partial}{\partial t} (\rho f) + \nabla \cdot (\rho f u) = \\
\frac{\partial}{\partial t} (\rho f) + (u \cdot \nabla) (\rho f) + (\rho f) \nabla \cdot u = \\
\frac{\partial}{\partial t} (\rho f) + (\rho f) \nabla \cdot u = \\
\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + \rho f \nabla \cdot u = \\
\rho \frac{\partial f}{\partial t} = f \frac{\partial \rho}{\partial t} + \rho f \nabla \cdot u = \\
\rho \frac{\partial f}{\partial t} \checkmark
\]
\[ \rho \frac{D}{Dt} \left[ u^2 + \frac{u^2 u_{ij} u_{ij}}{2} \right] = -V^\alpha \left( \rho u \right) + V^\alpha \left[ u^2 \frac{\partial}{\partial x_j} \right] \]

- \[ V^\alpha q + \rho g \]

New subtract off the equation for the mechanical kinetic energy which has no new information.

\[ u^\alpha \rho \frac{Du}{Dt} = u^\alpha \left[ -V \rho + pg + \frac{\partial}{\partial x_j} \right] \]

\[ \rho \frac{D}{Dt} \left( \frac{u^2 u_{ij}}{2} \right) = -u^\alpha V \rho + \rho g \cdot u + u^\alpha \left[ V \frac{\partial}{\partial x_j} \right] \]

\[ \nabla_j \left[ u_i \nabla_j u^i \right] - u_i \left[ \nabla_j \nabla_i u^i \right] = \]

\[ \left( \nabla_j u^i \right) \nabla_i u^j + u_i \left( \nabla_j \nabla_i u^i \right) - u_i \left( \nabla_j \nabla_i u^j \right) \]

\[ = \nabla_j \left( \nabla_i u^i \right) = T_{ij} \cdot \nabla_i u^i \]

\[ \nabla \cdot \cdot = \nabla \cdot \rho \nabla u \]
After the subheading...

\[ \rho \frac{D \mathbf{u}}{Dt} = -\rho \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q} + \mathbf{J} : \mathbf{\tau} \]

is the thermal energy equation.

Now, let's think about surface forces again.

\[ F_s \] (per unit volume)

Also, heat flux vector:

\[ \mathbf{q} = -k \nabla T \] (experimental fact for air purposes)

\[ \rho \frac{D \mathbf{u}}{Dt} = -\rho \nabla \cdot \mathbf{u} + \mathbf{J} \cdot (k \nabla T) + \mathbf{J} : \mathbf{\tau} \]
We could define

$$F_s = \nabla \cdot \sigma = -\nabla \rho + \nabla \cdot I$$

with $$\sigma = -\rho I + I$$

$\rho$ is the compressive normal stress at any point in a fluid at rest.

$I$ gives additional tensile normal and tangential stresses at a point in a fluid in motion.

$I$ symmetry follows from conservation of angular momentum.

For a Newtonian fluid

$$I = \lambda \text{Tr}(\sigma)I + \mu \sigma$$

$$
\sigma = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]

is the symmetrical part of \frac{1}{2} strain rate.

and \text{Tr} (\sigma) = \nabla \cdot \mathbf{u}$$
* \( \lambda, \mu \) can be different [think of adding purely compressive additional stresses]

* The sum of all the normal stresses:

\[
\text{Tr}(\mathbf{e}) = -3p + 3\lambda \text{Tr}(\mathbf{s}) + 2\mu \text{Tr}(\mathbf{s})
\]

\[
\rho = \frac{1}{3} \text{Tr}(\mathbf{e}) = \left( \frac{2}{3} \mu + \lambda \right) \mathbf{V} \cdot \mathbf{u}
\]

The quantity \( \left( \frac{2}{3} \mu + \lambda \right) \) is measured to be significantly different from zero only when compressibility effects are very strong, e.g., when shock waves are present.

**Stokes assumption:**

\( \lambda = -\frac{2}{3} \mu \Rightarrow \)

\[
\mathbf{e} = \rho \mathbf{I} + 2\mu \mathbf{S} - \frac{2}{3} \mu (\mathbf{V} \cdot \mathbf{u}) \mathbf{I}
\]

\[
\mathbf{F} = \nabla \cdot \mathbf{e} = -\nabla \left[ \rho + \frac{2}{3} \mu (\mathbf{V} \cdot \mathbf{u}) \right] + \nabla \cdot (2\mu \mathbf{S})
\]

\( \mu = \text{coefficient of viscosity} \)
Slightly revised system for a compressible, non-Newtonian fluid using Stokes' assumption
\[ \lambda = -\frac{2}{3} \mu \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \quad \text{(mass)} \]

\[ \rho \frac{D\mathbf{u}}{Dt} = -\rho \nabla p + \nabla \cdot \mathbf{T} \quad \text{(momentum)} \]

\[ \rho \frac{D\hat{\mathbf{u}}}{Dt} = -\rho \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{b} + \mathbf{T} \cdot \nabla \mathbf{u} \quad \text{(energy)} \]

\[ = -\rho \nabla \cdot \mathbf{u} + \mathbf{T} \cdot (k \nabla T) + \mathbf{T} \cdot \nabla \mathbf{u} \]

\[ \mathbf{T} = 2\mu \frac{\mathbf{D}}{Dt} - \frac{2}{3} \mu (\nabla \cdot \mathbf{u}) \mathbf{I} \]

\[ \mathbf{S} = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \]

\( \mu = \text{coefficient of viscosity} \)

\( k = \text{coefficient of thermal conductivity} \)

\( \rho = \rho(q, T) \quad \text{equation of state} \)

\( \rho = \rho(T) \quad \text{ideal gas} \)

\( \hat{\mathbf{u}} = \hat{\mathbf{u}}(T) \quad \hat{\mathbf{u}} = c_v T \quad \text{see notes} \)
So for an ideal gas:

\[ \rho \frac{D}{Dt} (E + T) = -\rho \nabla \cdot \mathbf{u} + \nabla \cdot (k \nabla T) + \mathbf{F} \otimes \mathbf{u} \]

and in general \( -\rho \nabla \cdot \mathbf{u} = \rho \rho \frac{D}{Dt} (\text{mass}) \)

\[ \Rightarrow 6 \text{ equations, 6 unknowns} \]

\[ u, p, \rho, T \quad \text{mass} \quad \text{3 momentum} \]

\[ \text{energy} \quad \text{1 energy} \quad \text{1 state} \]

Fluid properties \( u, k \) Thermodynamic quantities

\[ u = u(p, T) \quad k = k(p, T) \]

The Boussinesq Approximation

Now assume \( p \) depends mainly on \( T \)

[Neglect dependence on \( p \) and salt ]

\[ -\rho \nabla \cdot \mathbf{u} = \frac{\partial p}{\partial t} - \frac{\partial p}{\partial T} \quad \frac{\partial T}{\partial t} = -p \times \frac{\partial T}{\partial t} \]
with \( \alpha = -\frac{1}{\bar{c}_p} \frac{\partial P}{\partial T} \) the thermal expansion coefficient: \( [\alpha] = \left[ \frac{1}{T} \right] \)

Using the ideal gas law \( p = \frac{\rho RT}{V_0} \)

\[
\alpha = -\frac{1}{\bar{c}_p} \frac{\partial P}{\partial T} \rho = -\frac{1}{\bar{c}_p} \left( \frac{-\rho}{RT^2} \right) = \frac{1}{T}
\]

So \( -\rho \frac{\partial u}{\partial T} \alpha = -\frac{\rho RT}{T} \frac{DT}{Dt} = -\rho (c_p - c_v) \frac{DT}{Dt} \)  

\[ \uparrow \]  

see notes

Now neglecting viscous heating:

\[
\rho \frac{D}{Dt} (c_v T) \approx \nabla \cdot (k \nabla T) - \rho \frac{c_p - c_v}{c_p} \frac{DT}{Dt}
\]

or

\[
\rho \frac{D}{Dt} (c_p T) \approx \nabla \cdot (k \nabla T)
\]

treating \( c_p, k \) as constants

\[
\rho c_p \frac{DT}{Dt} = k \nabla^2 T
\]
\[ \frac{\partial p}{\partial t} = \frac{1}{\rho} \frac{\partial}{\partial t} \left( \frac{1}{\rho} \frac{\partial p}{\partial t} \right) + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial p}{\partial \rho} \right) \]

Then, is the ideal gas \( p = \rho RT \) ??

\[ c = \left( \frac{\partial p}{\partial \rho} \right)_s \sqrt{\frac{1}{g}} \quad \text{s - entropy} \]

Ignoring the difference between constant T, constant S (think about this...)

Then \( \frac{\partial p}{\partial \rho} \approx \frac{1}{c^2} \)

Dry air at 20°C \( c = 343 \text{ m/s} \)

Water \( c \approx 1500 \text{ m/s} \)

So, this is a time scale approximation:

Keeping compressibility effects for buoyancy only, not taking into account those sound waves radiating away.
e.g. in the atmosphere & blobs of fluid
move up \Rightarrow \text{they can encounter lower pressure}
\Rightarrow \text{they expand and cool}

* To justify neglecting viscous heating, let's
compute the terms \( \frac{\partial \hat{u}}{\partial t} \) and

\( \hat{I} : \nabla \hat{u} \) (spatial vorticity)

\( \hat{I} : \nabla \hat{u} \) scales as \( \rho \mu U^2 \frac{L^2}{V_x} \)

\( \rho \frac{\partial \hat{u}}{\partial t} \) \hspace{1cm} \rho_0 c \Delta T \frac{U_x}{V_x} 

\Rightarrow \frac{U_x}{\Delta T} \frac{V_x}{V_y} \text{ with a}

\text{typical liquid value } \frac{V_x}{c_p} \sim 10^{-9} \text{ s K} \left( \frac{1}{c_p \text{ typical}} \right)

\Rightarrow \frac{U_x}{\Delta T} \text{ very large for this term to be important}