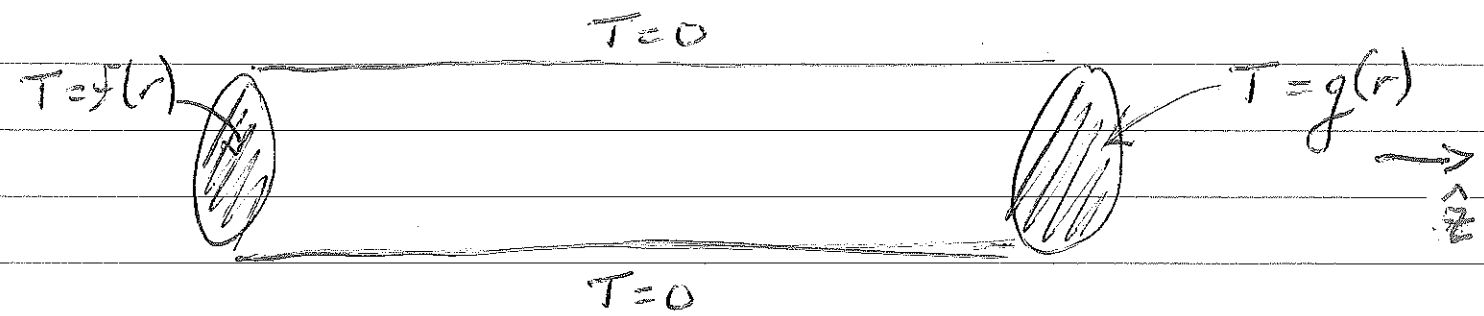
Math 322

Another physical context (simpler, Bessel of order zero)

$\frac{\partial T}{\partial t} = K \nabla^2 T$ in cylindrical coordinates and the steady-state limit



$T(0, z)$ bounded $\Rightarrow f(r), g(r)$ bounded as $r \to 0$ and for consistency we take $f(r) = g(r) = 0$ at $r = 1$

In cylindrical coordinates ; steady state

$$\cancel{\frac{\partial T}{\partial t}} = K \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) T$$

If we also consider axisymmetric solutions :

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) T = 0$$

Now separation of Variables $T(r,z) = \varphi(r)\,\psi(z)$

$$\Rightarrow \quad \frac{1}{\varphi}\left[\varphi'' + \frac{1}{r}\varphi'\right] = -\frac{\psi''}{\psi} = -\lambda$$

$$\Rightarrow \quad \varphi''(r) + \frac{1}{r}\varphi'(r) + \lambda\varphi(r) = 0$$ Bessel of order zero

interior $\quad \varphi(0)$ bounded

sidewalls $\quad \varphi(1) = 0$

$$\psi''(z) - \lambda\psi(z) = 0$$

endwalls $\quad T(r,0) = f(r) = \varphi(r)\,\psi(0)$

$$T(r,1) = g(r) = \varphi(r)\,\psi(1)$$

Bessel of order zero in SL form

$$\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = -\lambda r\varphi(r)$$

$$\varphi(0) \text{ bounded}, \quad \varphi(1) = 0$$

How do we solve Bessel's Eqn? (a 2^{nd}-order linear, non-constant coefficient, homogeneous ODE)

For convenience, lets scale out λ:

Let $x = \sqrt{\lambda}\, r$, $\quad \frac{dx}{dr} = \sqrt{\lambda}$

$$\frac{dY}{dr} = \frac{dY}{dx}\frac{dx}{dr} = \sqrt{\lambda}\,\frac{dY}{dx}$$

$$\frac{d^2Y}{dr^2} = \frac{d}{dr}\left(\sqrt{\lambda}\,\frac{dY}{dx}\right) = \sqrt{\lambda}\,\frac{d^2Y}{dx^2}\frac{dx}{dr} = \lambda\frac{d^2Y}{dx^2}$$

$$\lambda\frac{d^2Y}{dx^2} + \frac{\sqrt{\lambda}}{x}\sqrt{\lambda}\,\frac{dY}{dx} + \lambda Y = 0 \qquad Y = Y(x)$$

$$\Rightarrow \boxed{y'' + \frac{1}{x}y' + y = 0 \qquad \begin{array}{l} y(0) \text{ bounded} \\ y(\sqrt{\lambda}) = 0 \end{array}}$$

This type of equation has a Frobenius series soln.

$$y(x) = x^{\alpha}\sum_{n=0}^{\infty} a_n x^n \qquad (\text{about } x_0 = 0)$$

So-called "standard form" (not "SL" Form):

$$y'' + \tilde{p}(x)\, y' + \tilde{q}(x)\, y = 0$$

Homogeneous, linear, non-constant coefficient

* For $\tilde{p}(x)$, $\tilde{q}(x)$ analytic in a neighborhood of x_0 in the complex plane (x_0 is an ordinary point)

$\Rightarrow$ $\tilde{p}(x)$, $\tilde{q}(x)$ have Taylor Series expansions

$$\tilde{p}(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n \qquad n \text{ integer}$$

$$\tilde{q}(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n \qquad n \text{ integer}$$

Then the solution has a convergent Taylor series expansion

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad n \text{ integer}$$

with radius of convergence at least as large as the distance from x_0 to the nearest singularity in the complex plane

* For x_0 a regular singular point:

$$(x-x_0)\,\tilde{p}(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n \qquad n \text{ integer}$$

$$(x-x_0)^2\,\tilde{q}(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n \qquad n \text{ integer}$$

Then there is at least one solution of the form

$$y(x) = (x-x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad \alpha \text{ real}, \; n \text{ integer}$$

$$= (x-x_0)^{\alpha} A(x)$$

where $A(x)$ is analytic at $x = x_0$, with Taylor series expansion at least as large as the distance to the nearest singularity in the complex plane.

Back to Bessel of order zero:

"standard form" $y'' + \frac{1}{x} y' + y = 0$

$y(0), y'(0)$ bounded $\qquad y(\sqrt{\lambda}) = 0$

$X_0 = 0$ is a regular singular point:

$x \tilde{p}(x) = x \cdot \frac{1}{x} = \underline{1}$ analytic

$x^2 \tilde{q}(x) = x^2$ analytic

Therefore we look for a solution

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$$

$$y'(x) = \sum_{n=0}^{\infty} (\alpha+n) a_n x^{n+\alpha-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1) a_n x^{n+\alpha-2}$$

and plug into $x^2 y'' + x y' + x^2 y = 0$

(easier if we don't need to divide)

$$\sum_{n=0}^{\infty} a_n(\alpha+n)(\alpha+n-1)\,x^{\alpha+n} + \sum_{n=0}^{\infty} a_n(\alpha+n)\,x^{\alpha+n}$$

$$+ \sum_{n=0}^{\infty} a_n\,x^{\alpha+n+2} = 0$$

Equate like powers of x:

x^{α} : $a_0\alpha(\alpha-1) + a_0\alpha = 0$

For $a_0 \neq 0 \Rightarrow \alpha^2 = 0 \Rightarrow \alpha = 0$ repeated root

$x^{\alpha+1}$: $a_1(\alpha+1)\alpha + a_1(\alpha+1) = 0$

$a_1(\alpha^2 + 2\alpha + 1) = 0 \Longrightarrow a_1 = 0$

$x^{\alpha+2}$: $a_2(\alpha+2)(\alpha+1) + a_2(\alpha+2) + a_0 = 0$

$x^{\alpha+3}$: $a_3(\alpha+3)(\alpha+2) + a_3(\alpha+3) + a_1 = 0$

etc. $\Longrightarrow$ all odds zero and

$$a_n(\alpha) = -\frac{a_{n-2}}{(\alpha+n)^2} \qquad n \geq 2$$

with $\alpha = 0 \Rightarrow$

$$y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \right] = a_0 J_0(x)$$

The Bessel function of the 1st kind of order zero

Since ~~our~~ our eqn. is 2^{nd}-order, we need to find a 2^{nd} linearly independent solution

* use Reduction of Order, or

* By analogy with Euler equations, try

$$y_2(x) = J_0(x) \ln x + x^{\alpha} \sum_{n=0}^{\infty} b_n x^n$$

different coefficients

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m \, x^{2m}}{2^{2m} (m!)^2}$$

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

Back to ar pipe flow problem, the solution for the radial dipendence is

$$\varphi(r) = C_1 J_0(\sqrt{\lambda}\, r) + C_2\, y_2(\sqrt{\lambda}\, r)$$

since $x = \sqrt{\lambda}\, r$

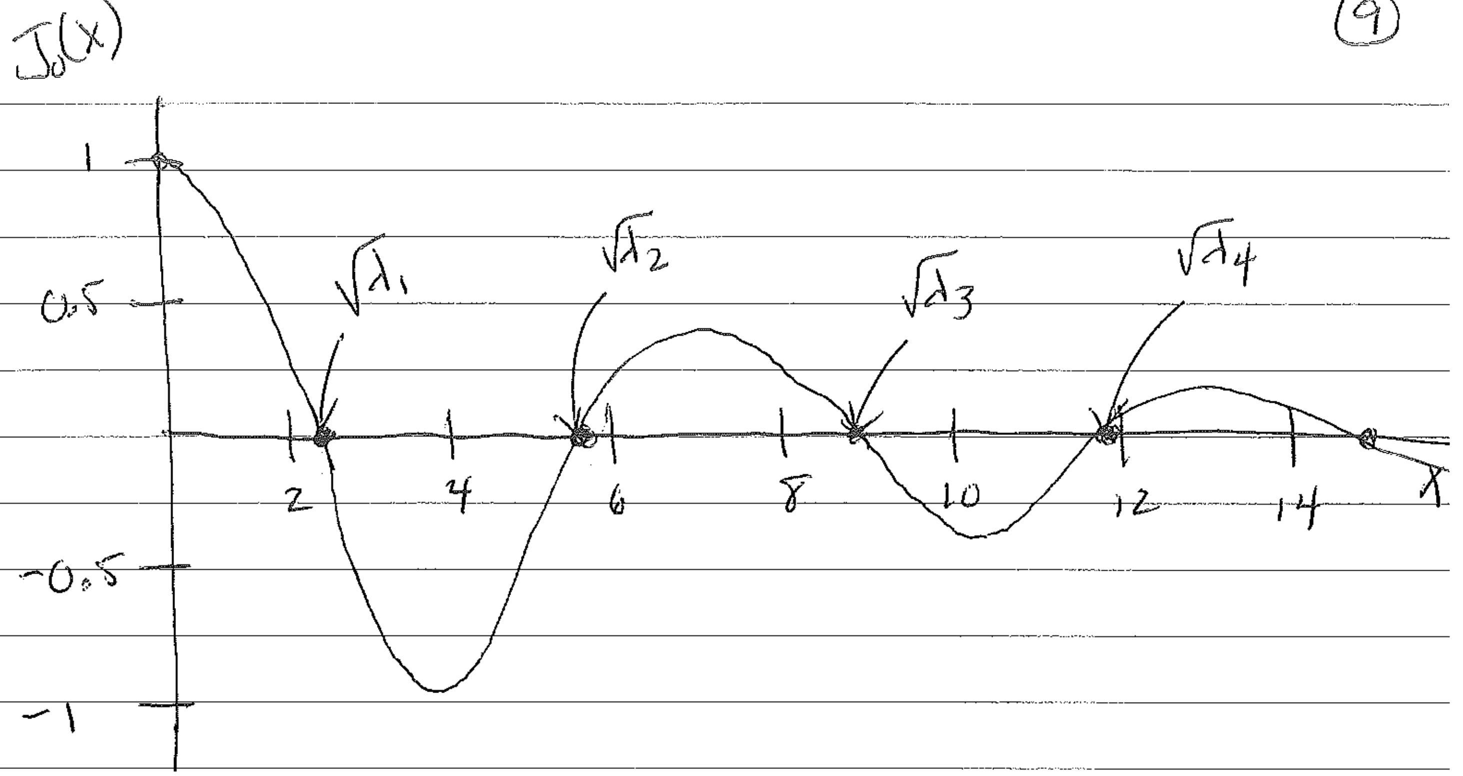
But $y_2(\sqrt{\lambda}\, r)$ has a factor of $\ln(\sqrt{\lambda}\, r)$ so it is not bunded at $r = 0$

$$\Rightarrow \quad C_2 = 0$$

$$\Rightarrow \quad \varphi(r) = C_1 J_0(\sqrt{\lambda}\, r) \quad \text{with } \varphi(1) = 0$$

and the bundary condition $\varphi(1) = 0$ determines the values of λ:

$$\varphi(1) = J_0(\sqrt{\lambda}) = 0$$



$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

the eigenvalues

$$\lim_{n \to \infty} \lambda_n = \infty$$

$$J_0(x) \approx \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \pi/4\right) \quad \text{as} \quad x \to \infty$$

For each λ_n, there is an eigenfunction

$\varphi_n(r) = J_0(\sqrt{\lambda_n}\, r)$ that satisfies the

ODE <u>and</u> the boundedness condition at $r=0$

<u>and</u> $\varphi_n(1) = 0$