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Review (Brief) 2^{nd}-order, linear, homogeneous ODEs in "standard form":

$$y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0$$

with non-constant coefficients.

[A] If $\tilde{p}(x)$, $\tilde{q}(x)$ are analytic in a neighborhood of x_0 in the complex plane, then x_0 is called an "ordinary point"

"analytic" means that there is a convergent Taylor Series expansion

$$\tilde{p}(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n \qquad n \text{ integer}$$

$$\tilde{q}(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

with a finite radius of convergence.

Then the solution also has a convergent Taylor series:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{with radius of}$$

convergence at least as large as the distance from x_0 to the nearest singularity in the complex plane

$\boxed{B}$ For x_0 a "regular singular point":

$$(x-x_0)\,\tilde{p}(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$$

$$(x-x_0)^2\,\tilde{q}(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

with finite radius of convergence. Then $\exists$ at least one solution of the form

$$y(x) = (x-x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x-x_0)^n = (x-x_0)^{\alpha} A(x)$$

α real, n integer

where $A(x)$ is analytic at $x=x_0$, with radius of convergence at least as large as the distance from x_0 to the nearest singularity in the complex plane

$\boxed{C}$ Anything else is an "irregular singular point"

Try $y(x) = \exp\left[B(x-x_0)\right] \sum_{n=0}^{\infty} a_n (x-x_0)^{\gamma n}$

γ real, $B(x)$ a function, not guaranteed to converge!!

$\boxed{B}$ Analogy with Euler/Equidimensional

$ax^2y'' + bxy' + cy = 0$ has solutions

$y(x) = x^r$ with r given by

$a(r-1)r + br + c = 0$

standard form:

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0$$

Now consider

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

where $b(x)$, $c(x)$ are infinitely differentiable with Taylor series (convergent). This situation looks very similar to Euler, so it makes sense to try powers of x multiplied by a "nice" function

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n = x^\alpha A(x) \quad \text{where}$$

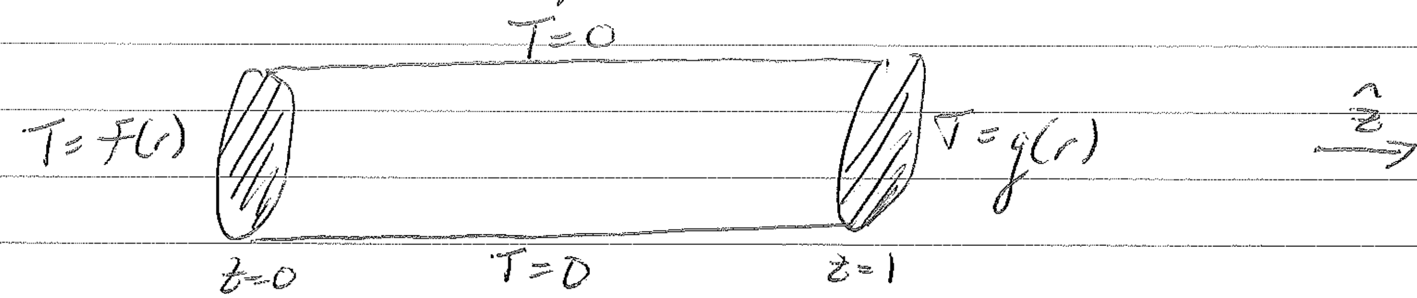
$A(x)$ has a convergent Taylor series

Now we just need to extend to "regular singularity at x_0 instead of zero" $\Rightarrow$

$$y(x) = (x - x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$\{$ For Euler, $x_0 = 0$ and $A(x) = 1$ $\}$

Back to our Bessel / Heat Conduction Problem



$$\frac{\partial T}{\partial t} = K\nabla^2 T \qquad 0<r<1, \; 0<z<1, \; -\pi<\theta\le\pi, \; t>0$$

axisymmetric, steady state: $T(r,z) = \varphi(r)\psi(z)$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\varphi(r)\psi(z) = 0$$

$$\Longrightarrow \boxed{\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = -\lambda r\varphi \qquad \begin{array}{l}\varphi(0) \text{ bounded} \\ \varphi(1) = 0\end{array}}$$

Bessel of order zero

$$\frac{d^2\psi}{dz^2} = \lambda\psi(z)$$

$$T(r,0) = f(r) = \varphi(r)\psi(0)$$
$$T(r,1) = g(r) = \varphi(r)\psi(1)$$

Let $x = \sqrt{\lambda}\, r \Rightarrow$

$$\frac{d}{dx}\left(x \frac{dy}{dx}\right) = xy \qquad y(0) \text{ bounded} \;;\; y(\sqrt{\lambda}) = 0$$

SL Form

Standard Form $\quad y'' + \frac{1}{x} y' + y = 0$

$x_0 = 0$ is RSP ; Try $y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n$

Plug into non-singular form (easier):

$$x^2 y'' + xy' + x^2 y = 0$$

$$\sum_{n=0}^{\infty} (\alpha+n-1)(\alpha+n) a_n x^{\alpha+n}$$

$$+ \sum_{n=0}^{\infty} (\alpha+n) a_n x^{\alpha+n} + \sum_{n=0}^{\infty} a_n x^{\alpha+n+2} = 0$$

Equate the coefficient of each power of x to zero (linear independence of powers of x)

$\Rightarrow \quad a_0$ arbitrary, $\alpha = 0 \qquad \{x^{\alpha}\}$

$a_1 = 0 \qquad \{x^{\alpha+1}\}$

$$a_2 = \frac{-a_0}{\left[(\alpha+1)(\alpha+2) + (\alpha+2)\right]} \qquad \{x^{\alpha+2}\}$$

$$a_2(\alpha+2)(\alpha+1) + a_2(\alpha+2) + a_0 = 0 \qquad \{x^{\alpha+2}\}$$

$$a_3(\alpha+3)(\alpha+2) + a_3(\alpha+3) + a_1 = 0 \qquad \{x^{\alpha+3}\}$$

etc. $\Rightarrow$ all odds are zero and

$$a_n(\alpha) = -\frac{a_{n-2}}{(\alpha+n)^2} \qquad n \geq 2$$

the recursion relation.

We can also change the center:

$$\sum_{n=0}^{\infty} \left[(\alpha+n-1)(\alpha+n) + (\alpha+n)\right] a_n x^{\alpha+n}$$

$$+ \sum_{m=2}^{\infty} a_{m-2} x^{\alpha+m} = 0$$

x^{α}: $\alpha = 0$, a_0 arbitrary

$x^{\alpha+1}$: $a_1 = 0$

$$\sum_{n=2}^{\infty} \left[(\alpha+n)^2 a_n + a_{n-2}\right] = 0$$

$$a_n = \frac{-a_{n-2}}{(\alpha+n)^2} \qquad n \geq 2$$

with $\alpha = 0 \Rightarrow$

$$y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \right]$$

$= a_0 J_0(x)$ 1st kind of order zero

But notice that we've only generated one solution because $\alpha^2 = 0 \Rightarrow \alpha = 0$ a repeated root

This method fails in some cases to generate 2 linearly independent solution

What do we do? Reduction of Order, or in this case we can "guess" a solution based on analogy with Euler's Equation

$$y_2(x) = J_0(x)\ln x + x^{\alpha} \sum_{n=0}^{\infty} b_n x^n$$

different coefficients

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m x^{2m}}{2^{2m}(m!)^2}$$

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$$

2nd kind of order zero

So we arrive at

$$y(x) = C_1 J_0(x) + C_2 y_2(x) \quad \text{or}$$

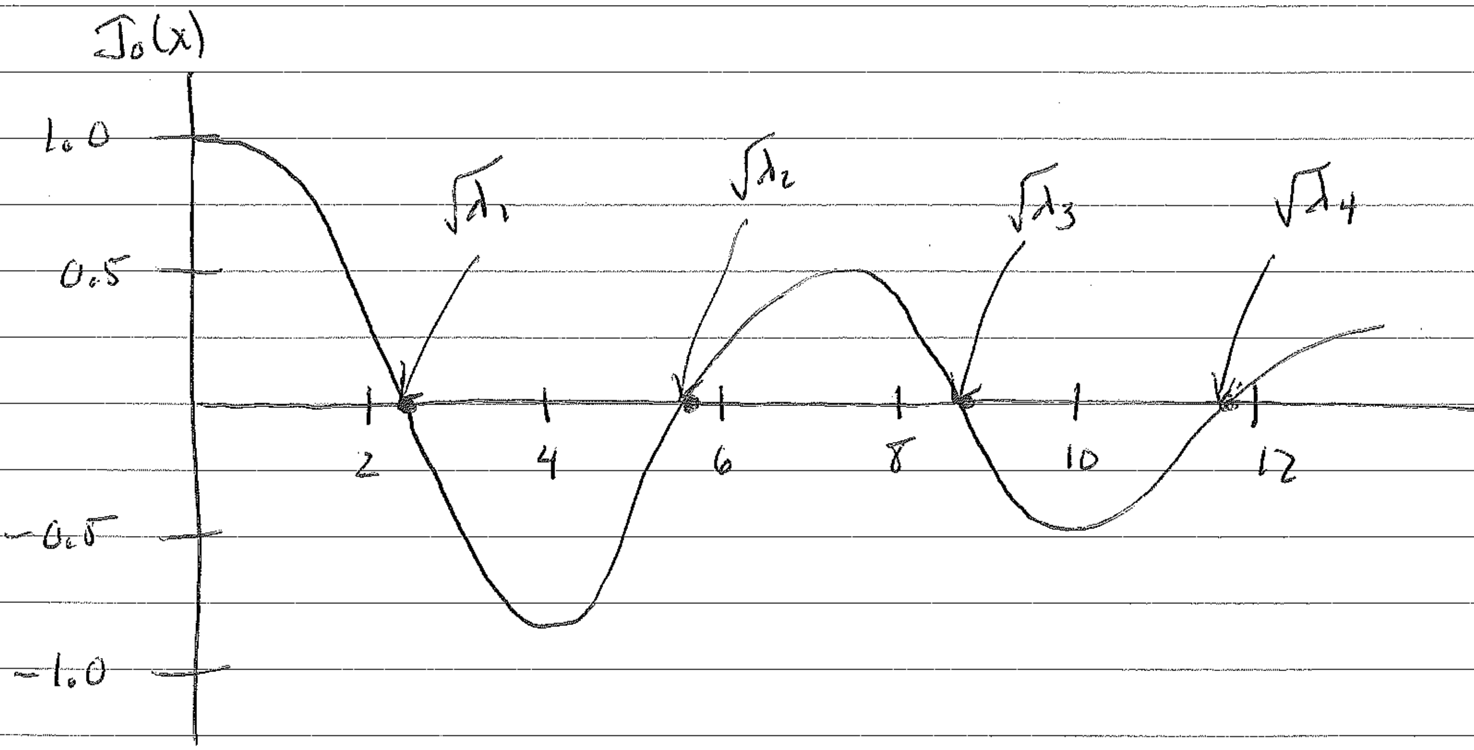
$$\varphi(r) = C_1 J_0(\sqrt{\lambda}\, r) + C_2 y_2(\sqrt{\lambda}\, r)$$

and now we need to satisfy

* $\varphi(r=1) = 0$
* $\varphi(r=0)$ bounded $\Longrightarrow C_2 = 0.$

Thus $\varphi(r) = C_1 J_0(\sqrt{\lambda}\, r)$

and $\varphi(1)=0$ determines the values of λ:



$$\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$$

For each λ_n, there is an eigenfunction $\varphi_n(r)$:

$\varphi_n(r) = J_0(\sqrt{\lambda_n}\, r)$ that satisfies the ODE and boundedness at $r=0$ and $\varphi_n(1) = 0$

Since the problem is linear, we can sum solutions

$$\varphi(r) = \sum_{n=1}^{\infty} C_n J_0(\sqrt{\lambda_n}\, r)$$

Since the problem is linear, we can sum solutions

$$\varphi(r) = \sum_{n=1}^{\infty} C_n J_0(\sqrt{\lambda_n}\, r) \qquad \text{general solution}$$

Why do we need them all? Answer: to be able to represent any endwall boundary conditions $f(r)$, $g(r)$ in problem # 2

$\boxed{\#2}$ $\quad \psi''(z) - \lambda \psi(z) = 0$

$$\psi(0)\varphi(r) = f(r) \qquad \psi(1)\varphi(r) = g(r)$$

For each λ_n: $\quad \psi_n''(z) - \lambda_n \psi_n(z) = 0 \implies$

$$\psi_n(z) = A_n \exp(\sqrt{\lambda_n}\, z) + B_n \exp(-\sqrt{\lambda_n}\, z)$$

Then $T_n(r,z) = \varphi_n(r)\psi_n(z)$ and summing

$$T(r,z) = \sum_{n=1}^{\infty} C_n J_0(\sqrt{\lambda_n}\, r)\left\{A_n e^{\sqrt{\lambda_n}\, z} + B_n e^{-\sqrt{\lambda_n}\, z}\right\}$$

$$= \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_n}\, r)\left\{a_n e^{\sqrt{\lambda_n}\, z} + b_n e^{-\sqrt{\lambda_n}\, z}\right\}$$

Now satisfy the endwall conditions $\implies$

(a) $T(r,0) = f(r) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_n}\, r) \{ a_n + b_n \}$

(b) $T(r,1) = g(r) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_n}\, r) \{ a_n e^{\sqrt{\lambda_n}} + b_n e^{-\sqrt{\lambda_n}} \}$

Finally we use orthogonality of Bessel functions

$$\int_0^1 J_0(\sqrt{\lambda_n}\, r) J_0(\sqrt{\lambda_m}\, r)\, r\, dr = \Omega\, \delta_{nm}$$

Multiply (a) by $r J_0(\sqrt{\lambda_m}\, r)$ and integrate from 0 to 1

$$\Rightarrow \quad a_m + b_m = \frac{1}{\Omega} \int_0^1 r J_0(\sqrt{\lambda_m}\, r) f(r)\, dr$$

Multiply (b) by $r J_0(\sqrt{\lambda_m}\, r)$ and integrate from 0 to 1

$$\Rightarrow \quad a_m e^{\sqrt{\lambda_m}} + b_m e^{-\sqrt{\lambda_m}} = \frac{1}{\Omega} \int_0^1 r J_0(\sqrt{\lambda_m}\, r) g(r)\, dr$$

2 eqn.s for 2 unknowns a_m, b_m

This is a formal solution to our heat conduction problem

$$T(r,z) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_n}\, r)\left\{ a_n e^{\sqrt{\lambda_n}\, z} + b_n e^{-\sqrt{\lambda_n}\, z} \right\}$$

$$a_n + b_n = \frac{1}{\Omega} \int_0^1 r\, J_0(\sqrt{\lambda_n}\, r)\, f(r)\, dr$$

$$a_n e^{\sqrt{\lambda_n}} + b_n e^{-\sqrt{\lambda_n}} = \frac{1}{\Omega} \int_0^1 r\, J_0(\sqrt{\lambda_n}\, r)\, g(r)\, dr$$

What elements did we use from ODE theory that we should review?

<u>**At least**</u>:

* Series solutions about ordinary points, regular singular points, irregular singular points

* Reduction of Order

* Orthogonality of special functions arising from so-called Sturm-Liouville Problems