St problem \[ p \phi'' + q \phi = -10 \phi \quad a < x < b \]

with appropriate boundary conditions (regular, singular, periodic)

leads to the generalized eigenfunction expansion

\[ f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x) \quad a < x < b \]

For piecewise smooth functions \( f(x) \),

By orthogonality:

\[ \int_a^b f(x)\phi_n(x)\sigma(x)\,dx = \int_a^b a_n \phi_n^2(x)\sigma(x)\,dx \]

\[ \Rightarrow a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} \]

In practice, we usually need to use a finite

\[ f(x) \sim \sum_{n=1}^{M} a_n \phi_n(x) \]
and it is not clear that we should choose

\[ \alpha_n = a_n \]? Is this the choice that will
give the "best" approximation for \( f(x) \)
for any given truncation \( M \)? Do the "best"
values of \( \alpha_n \) change with \( M \)?

To define "best," let's first define the error. Again there are different measures of the error.

\[ E_1 = \max \left| f(x) - \sum_{n=1}^{M} \alpha_n d_n(x) \right| \]

Note that \( f(x) - \sum_{n=1}^{M} \alpha_n d_n(x) \) is a function
of \( x \); we find the largest in absolute value.

\[ E_2 = \int_a^b \left[ f(x) - \sum_{n=1}^{M} \alpha_n d_n(x) \right]^2 \sigma(x) \, dx \]

the mean-square error turns out to have
nec properties
The choice $\alpha_n = \alpha_0$ minimizes the mean square error for any choice of $M$.

To see that this might be the case, we would require

$$\frac{dE_i}{d\alpha_i} = 0 \quad i = 1, 2, 3, \ldots, M$$

Now the unknowns are the $\alpha_i$'s and we want a minimum in the space of $\alpha_i$'s.

This condition is not enough to ensure a min (could be a max or a saddle), but let's check

$$\frac{dE_i}{d\alpha_i} = \int_a^b 2 \left[ F(x) - \sum_{n=1}^M \alpha_n \phi_n(x) \right] \phi_i^*(x) \sigma(x) \, dx = 0$$

For each $i = 1, 2, 3, \ldots, M$

$M$ equations for $M$ unknowns,

$$\int_a^b 2 \int_a^b \phi_i^*(x) \sigma(x) \, dx = \int_a^b 2 \sum_{n=1}^M \alpha_n \phi_n(x) \phi_i^*(x) \sigma(x) \, dx$$

Now use orthogonality
\[
\int_a^b f(x) \, d\psi_i(x) \sigma(x) \, dx = \int_a^b a_i \phi_i^2(x) \sigma(x) \, dx
\]

Now it doesn't matter if we call it something else \((m, n, \text{etc.})\).

\[a_i = \frac{\int_a^b f(x) \phi_i(x) \sigma(x) \, dx}{\int_a^b \phi_i^2(x) \sigma(x) \, dx}\]

\[a_n = a_n = \frac{\int_a^b f(x) \phi_n(x) \sigma(x) \, dx}{\int_a^b \phi_n^2(x) \sigma(x) \, dx}\]

So the chain \(a_n = a_n\) is a good candidate for minimizing the mean square error, but we don't yet have a proof.

Proof may derivative is harder than proof using complete-the-square.
Start again with

\[ E_2 = \int_a^b \left[ f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x) \, dx \]

\[ = \int_a^b \sum_{n=1}^{M} \left[ f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x) \, dx \]

The third term:

\[ \int_a^b \sum_{n=1}^{M} \alpha_n \phi_n(x) \sum_{n=1}^{M} \alpha_n \phi_n(x) \sigma(x) \, dx \]

\[ = \sum_{n=1}^{M} \sum_{l=1}^{M} \int_a^b \alpha_n \alpha_l \phi_n(x) \phi_l(x) \sigma(x) \, dx \]

\[ = \sum_{n=1}^{M} \int_a^b \alpha_n \phi_n(x) \phi_n(x) \sigma(x) \, dx \]

\[ = \int_a^b \sum_{n=1}^{M} \alpha_n^2 \phi_n^2(x) \sigma(x) \, dx \]

\[ E_2 = \int_a^b \sum_{n=1}^{M} \alpha_n^2 \phi_n^2 - \sum_{n=1}^{M} \alpha_n f \phi_n \sigma + \sum_{n=1}^{M} \alpha_n f \phi_n^2 \sigma \, dx \]

with the last 2 terms dependent on \( \alpha_n \).
Isolating the last 2 terms:

\[
\sum_{n=1}^{M} \left\{ \int_{a}^{b} \alpha_{n}^{2} \Phi_{n}^{2} \sigma \, dx - \int_{a}^{b} 2\alpha_{n} \Phi_{n} \sigma \, dx \right\} =
\]

\[
\sum_{n=1}^{M} \left[ \gamma_{n}^{2} \sigma - 2\gamma_{n} \beta \right]
\]

where \( \gamma = \int_{a}^{b} \Phi_{n} \sigma \, dx \), \( \beta = \int_{a}^{b} \Phi_{n} \sigma \, dx \)

\[
\sum_{n=1}^{M} \left[ \gamma_{n}^{2} \sigma - 2\gamma_{n} \frac{\beta}{\delta} \right] =
\]

\[
\sum_{n=1}^{M} \gamma_{n}^{2} \sigma - \frac{2}{\delta} \sum_{n=1}^{M} \gamma_{n} \beta^{2}
\]

Go back to the entire \( E_{2} \)

\[
E_{2} = \int_{a}^{b} F_{0}^{2} \, dx + \sum_{n=1}^{M} \gamma_{n}^{2} \sigma - \frac{2}{\delta} \sum_{n=1}^{M} \gamma_{n} \beta^{2}
\]

\[
\uparrow
\]

\[
\text{independent of } \alpha_{n}
\]

\[
\uparrow
\]

\[
\text{independent of } \sigma
\]

and recall that \( E_{2} \) by its original form

\[
E_{2} = \int_{a}^{b} \left[ f(x) - \sum_{n=1}^{M} \alpha_{n} \Phi_{n}(x) \right]^{2} \sigma(x) \, dx \geq 0
\]
To minimize $E_2$ with $\alpha_n$, set middle term = 0

\[ \implies \alpha_n = \frac{B^2}{\delta} = \frac{\int_a^b F \phi_n a\ dx}{\int_a^b \phi_n^2 a \ dx} \]

and the minimal error is

\[ E_2 = \int_a^b F^2 a\ dx - \sum_{n=1}^{M} \frac{\phi_n^2}{\delta} \]

\[ = \int_a^b F^2 a\ dx - \sum_{n=1}^{M} \alpha_n^2 a \]

\[ \geq 0 \]

Minimal error

What did we do?

* Original definition of $E_2$
* Expanded $E_2$ used orthogonality
* Completed the square
* Chose $\alpha_n$ to minimize $E_2$
For Fourier sine series with $\sigma(x) = 1$, $a = 0$, $b = 1$

$$a_n(x) = \sin \frac{\pi x}{L} \quad \frac{1}{L} \int_0^L \sin^2 \frac{\pi x}{L} \, dx = \frac{1}{2}$$

$$\Rightarrow \quad E_2 = \int_0^L f^2 \, dx = \frac{1}{2} \sum_{n=1}^{\infty} a_n^2$$

**Bessel's Inequality**

Since $E_2 \geq 0$ with $\sigma(x) > 0$,

$$\int_0^L f^2 \, dx \geq \sum_{n=1}^{\infty} a_n^2 \int_0^L \sigma^2 \, dx$$

We also know that

$$\sum_{n=1}^{\infty} a_n^2 \int_0^L \sigma^2 \, dx \text{ increases with } M$$

Since we are adding positive quantities

Therefore, the biggest value of the RHS is achieved for $M \to \infty$ and

**Parseval's Equality**

$$\int_0^L f^2 \, dx = \sum_{n=1}^{\infty} a_n^2 \int_0^L \sigma^2 \, dx$$
Parsell's Equality is important because it relates an integral "energy" of $f(x)$ to the generalized Fourier coefficients in physical space.

The energy to a Fourier space energy is

$$\int_{-\infty}^{\infty} f(x)^2 \, dx = \sum_{n=-\infty}^{\infty} a_n^2 \int_{-\infty}^{\infty} \left| \sin \omega \right|^2 \, d\omega$$

$$= \sum_{n=-\infty}^{\infty} a_n^2 \frac{\omega}{2} \left( \sin \frac{\omega}{2} \right)^2$$

$$= \sum_{n=-\infty}^{\infty} (a_n l)^2$$

where $l$ is the length squared of the sum of the squares of the components of $f(x)$ using the orthogonal basis $\sin \omega$.