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Higher-dimensional problems may lead to a generalized Sturm Liouville problem for spatial dependence:

$$\nabla \cdot (\rho \nabla \phi) + q\phi = -\lambda \sigma \phi$$

$$\left(a\phi + b \nabla \phi \cdot \hat{n}\right)\Big|_{\Omega} = 0$$

$\big|_{\Omega}$ means evaluated at the boundary

with similar theorems/results as in 1D (but not exactly the same)

For simplicity, lets consider $\rho = 1$, $q = 0$, $\sigma = 1$

$\Rightarrow$ Helmholtz Equation

$$\nabla^2 \phi + \lambda \phi = 0$$

$$a\phi + b \nabla \phi \cdot \hat{n} = 0$$

leading to Fourier series in Cartesian geometry;
Bessel in Cylindrical geometry;
Legendre in Spherical geometry

What are the key results, independent of geometry?

* The operator (and boundary conditions) is symmetric
* The eigenvalues / eigenfunctions are real
* Eigenfunctions corresponding to different eigenvalues are orthogonal
* There is a Rayleigh quotient
* Convergence of a generalized eigenfunction expansion

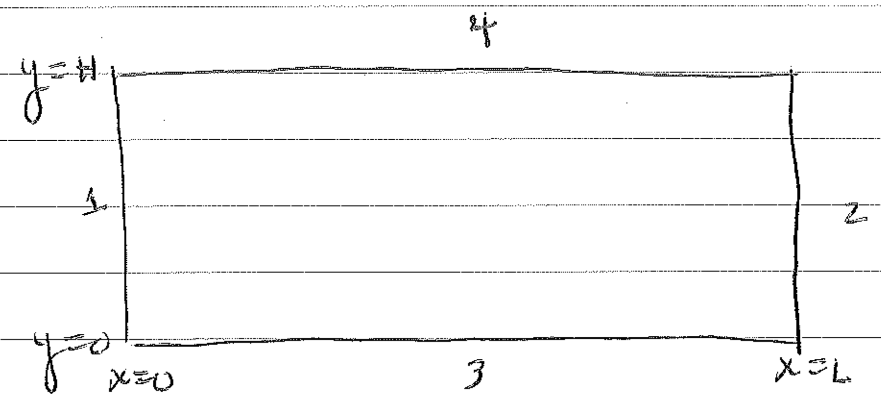
$$f(x) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x)$$

for $f(x)$ piecewise smooth

Examples

* Vibrating Rectangular Membrane (Fourier)
* Vibrating Circular Membrane (Bessel)
* Vibrating Sphere (Legendre)
  e.g. earthquakes $\boxed{7.10}$

Example 1 | Vibrating Rectangular Membrane with zero displacement on the boundary



$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad u = u(x,y) \qquad 0 < x < L$$
$$0 < y < H$$
$$t > 0$$

$u(x,y,0) = \alpha(x,y)$ initial displacement (interior)

$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$ initial velocity (interior)

$u(0,y,t) = 0$ side 1

$u(L,y,t) = 0$ side 2

$u(x,0,t) = 0$ side 3

$u(x,H,t) = 0$ side 4

Separation of Variables:

$$u(x,y,t) = f(x)\,g(y)\,h(t) = \phi(x,y)\,h(t)$$

$$\phi(x,y)\,\frac{d^2 h(t)}{dt^2} = c^2\,h(t)\,\nabla^2\phi(x,y)$$

$$\frac{1}{c^2 h(t)}\,\frac{d^2 h(t)}{dt^2} = \frac{1}{\phi(x,y)}\,\nabla^2\phi(x,y) = -\lambda$$

{ why do we choose the minus sign and expect $\lambda \geq 0$? }

$$\frac{d^2 h(t)}{dt^2} = -\lambda\,c^2\,h(t)$$

$$\nabla^2\phi(x,y) = -\lambda\,\phi(x,y)$$

Now $\phi(x,y) = f(x)\,g(y) \Rightarrow$

$$g(y)\,\frac{d^2 f(x)}{dx^2} + f(x)\,\frac{d^2 g(y)}{dy^2} = -\lambda\,f(x)\,g(y)$$

$$\frac{1}{f(x)}\,\frac{d^2 f(x)}{dx^2} + \frac{1}{g(y)}\,\frac{d^2 g(y)}{dy^2} = -\lambda$$

or $$\frac{1}{F}\frac{d^2F}{dx^2} = -\lambda - \frac{1}{g}\frac{d^2g}{dy^2} = -\mu$$

{why the $-$ sign with expectation $\mu \geq 0$?}

$$\frac{1}{F}\frac{d^2F(x)}{dx^2} = -\mu$$

$$\frac{1}{g}\frac{d^2g(y)}{dy^2} = -\lambda + \mu = -(\lambda - \mu)$$

Lets solve the 2 problems 1st, then return to the time dependence

$$\frac{d^2F(x)}{dx^2} = -\mu F(x) \qquad F(0) = 0 \quad F(L) = 0$$

$$\frac{d^2g(y)}{dy^2} = -(\lambda - \mu)\, g(y) \qquad g(0) = 0 \quad g(H) = 0$$

where the boundary conditions come from

$\phi(0,y) = 0$ side 1 $\qquad \phi(x,0) = 0$ side 3

$\phi(L,y) = 0$ side 2 $\qquad \phi(x,H) = 0$ side 4

$\boxed{\#1}$ $F'' = -\mu F$ $F(0) = F(L) = 0$

$F_n(x) = \sin\frac{n\pi x}{L}$, $\mu_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3 \ldots$

Now for each value of μ_n:

$\boxed{\#2}$ $g'' = -(\lambda - \mu_n)g$ $g(0) = g(H) = 0$

$g_m(y) = \sin\frac{m\pi y}{H}$ $\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2$ $m = 1, 2, 3 \ldots$

Finally $\phi_{nm}(x, y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$ $n = 1, 2, 3 \ldots$
$m = 1, 2, 3 \ldots$

$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ $\lambda_{nm} > 0$

$\boxed{\#3}$ $h'' = -\lambda_{nm} c^2 h = -\lambda_{nm} c^2 h$

$h(t) = A_{nm} \cos c\sqrt{\lambda_{nm}}\, t + B_{nm} \sin c\sqrt{\lambda_{nm}}\, t$

Putting everything together

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \left\{ A_{nm} \cos\left(c\sqrt{\lambda_{nm}}\, t\right) + B_{nm} \sin\left(c\sqrt{\lambda_{nm}}\, t\right) \right\}$$