Dispersive Wave Systems

* rotating flows, plasma flows, stratified flows
* Why are they different mathematically and physically from NS flows without waves

As an example, consider Boussinesq in a rotating frame \( \mathbf{R} = \mathbf{R} \hat{\mathbf{z}} \):

\[
\frac{D\mathbf{u}}{Dt} + \mathbf{\Omega} \times \mathbf{u} + \mathbf{N} \mathbf{\hat{z}} = -\nabla P + \nu \nabla^2 \mathbf{u}
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad \frac{D\mathbf{e}}{Dt} - \mathbf{N}(\mathbf{u} \cdot \mathbf{\hat{z}}) = \kappa \nabla^2 \theta
\]

\[
P = P_0 + \tilde{\rho}(x,t) \quad |\tilde{\rho}| \ll P_0
\]

\[
\tilde{\rho} = -b \mathbf{z} + \rho'(x,t) \quad \text{stably stratified} \quad \text{(density decreasing with altitude)}
\]

\[
\rho = \rho_{\text{hydro}}(z) + \rho'(x,t)
\]

\[
\rho' = \left( \frac{bP_0}{g} \right)^{\frac{1}{2}} \theta \quad \text{is rescaled so that}
\]

\( \theta \) has dimensions of velocity
\[ N = \left( \frac{g b}{k T_0} \right)^{1/2} \] is the buoyancy frequency

\[ P = p/p_0 \]

For atmospheric flow, it is convenient to use the potential temperature

\[ \Theta = T \left( \frac{p_0}{p} \right)^{R/C_P} \]

\[ C_P \text{ gas constant} \]

\[ \text{Cp specific heat} \]

which increases as altitude increases.

**Inviscid Limit**

Energy: \[ \frac{\nu \cdot u}{2} + \frac{E^2}{2} \]

Conserved globally

\[ u \cdot \text{momentum} + \Theta \times \text{energy} \Rightarrow \]

\[ \frac{d}{dt} \int_E dA = 0 \]
Conservation of Potential Vorticity Following Fluid Particles:
\[
\frac{D}{Dt} \left( \omega \cdot \nabla \rho \right) = 0
\]
\[
\omega_\alpha = 2 \pi \hat{z} + \omega
\]
\[
\omega = \nabla \times \mathbf{u}
\]

The Linear Inviscid Case:
\[
\frac{d\mathbf{u}}{dt} + \nabla \times \mathbf{u} + N \theta \hat{z} + \nabla \rho = 0
\]
\[
\nabla \cdot \mathbf{u} = 0, \quad \frac{d\theta}{dt} = -N (u \cdot \hat{z}) = 0
\]

Assuming periodic boundary conditions:
\[
\begin{bmatrix}
\mathbf{u}(x, t, j \hat{k}) \\
\theta(x, t, j \hat{k}) \\
p(x, t, j \hat{k})
\end{bmatrix}
= \begin{bmatrix}
\hat{\mathbf{u}}(k) \\
\hat{\theta}(k) \\
\hat{p}(k)
\end{bmatrix}
\exp \left[ ik \cdot x - \omega(k) t \right]
\]

Plugging in
\begin{align*}
-i\sigma \hat{\mathbf{u}} + \int \hat{\mathbf{z}} \times \hat{\mathbf{u}} + N\hat{\mathbf{z}} + i\kappa \hat{\rho} &= 0 \\
K \cdot \hat{\mathbf{u}} &= 0 \quad -i\sigma \hat{\theta} - N\hat{\mathbf{u}} \cdot \hat{\mathbf{z}} &= 0 \\
\text{with} \quad \mathbf{x} &= \begin{bmatrix} \hat{\mathbf{u}} & \hat{\mathbf{\theta}} & i\hat{\rho} \end{bmatrix}^T \\
A^* \mathbf{x} &= \lambda \mathbf{B} \mathbf{x} \\
A^* &= -A, \quad B^* = B, \quad \det(B) = 0 \\
\lambda &= i\sigma, \quad \text{last row of } B \text{ is zero} \\
H &= \text{conjugate transpose} \\
\text{"Easy" to show that} \\
\star \lambda \text{ is pure imaginary [i.e., real]} \\
\text{wave solution} \\
\star \text{Defining } \mathbf{v} = \begin{bmatrix} \hat{\mathbf{u}} & \hat{\mathbf{\theta}} \end{bmatrix}^T \quad \text{for } \lambda, \mathbf{v}, \\
\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 &= \mathbf{v}_1^* \mathbf{v}_2 = \mathbf{v}_2^* \mathbf{v}_1 = 0 \\
\text{orthogonality}
Neutral Stable!

Wave solutions are associated with the antisymmetric linear operator

The analysis/conclusions depend on $N^2$ positive $\Rightarrow$ stably stratified flow

Find the eigenvalues and eigenvecors

Method 1: Solve $AX = \lambda B X$

Method 2: $\nabla \times \nabla \times \text{momentum} \ldots$

Step 1: $k \times \text{momentum} \Rightarrow \text{(eliminates } \hat{\rho})$

$K \times \hat{\nu} = \frac{i}{\sigma} \left[ f_k k_z \hat{\nu} - N \hat{\epsilon} (k \times \hat{\nu}) \right]

f = 2 \Omega$

Step 2: $k \times (k \times \text{momentum}) \Rightarrow$

$i \sigma k^2 \hat{\nu} - F k_z (k \times \hat{\nu}) + N \hat{\epsilon} (k_z k \times \hat{\nu}) = 0$
Step 3: Use $k \times \hat{u} = 0$. From step 1 in step 2 result:

$$(\sigma^2 k^2 - \sigma^2 k^2 \hat{2}) \hat{u} + N \hat{G} (k \times \hat{2}) = 0$$

We can still use $k_0 \hat{u} = 0$ (mass) and $-i\sigma \hat{G} = N \hat{w}$ (energy).

Wave Modes: $\sigma \neq 0 \Rightarrow \hat{w} \neq 0$

Take $\hat{z} \neq 0$; use $-i\sigma \hat{G} = N \hat{w}$

$$\Rightarrow (\sigma^2 k^2 - \sigma^2 k^2 \hat{2}) \hat{w} + N^2 (k^2 - k^2 \hat{2}) \hat{w} = 0$$

and $\hat{w} \neq 0 \Rightarrow$

$$\sigma^+ (k) = \frac{1}{1} \left( \sigma^2 k^2 + N^2 k^2 \right)^{1/2} \left| k \right|$$
is the dispersion relation for the waves

Find the eigenmodes

$$\Phi^+ = \int \frac{d\xi}{\tilde{e}}$$

For \( \sigma^+ \) from \(*\star\)

$$\begin{bmatrix} \hat{w} \\ \hat{v} \\ \hat{\tilde{w}} \end{bmatrix} + \frac{i\omega F k_f k_z}{\sigma^+ k_h} \begin{bmatrix} -k_y \\ k_x \\ 0 \end{bmatrix} - \frac{\hat{\tilde{w}}}{\sigma^+ k_h} \begin{bmatrix} k_x k_2 \\ k_y k_2 \\ -k_h^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a thermal vector pick \( \hat{\tilde{w}} = -\frac{k_h}{\sqrt{2} |k|} \)

Then

$$\hat{\Theta} = \frac{-i N k_h}{\sqrt{2} \sigma^+ k_h}$$

$$\Phi^+ = \frac{1}{\sqrt{2} \sigma^+ k_h |k|}$$

$$k_h \neq 0$$

$$\begin{bmatrix} \sigma^+ k_x k_2 + i f k_y k_2 \\ \sigma^+ k_y k_2 - i f k_x k_2 \\ -k_h^2 \sigma^+ \\ -i N k_h^2 \end{bmatrix}$$
Singular if \( k_n = 0 \) so choose

\[
\hat{\phi} (0, 0, k_z) = \begin{bmatrix}
\frac{1}{2} (1 + i \text{sgn} (k_z)) \\
\frac{1}{2} (1 - i \text{sgn} (k_z)) \\
0 \\
0
\end{bmatrix}
\]

which have \( \hat{w} = 0 \), \( \hat{w}_z = 0 \), \( \sigma^\pm = \pm \frac{1}{\epsilon} \)

Also, "vertical modes" with \( \sigma = 0 \) \( \Rightarrow \hat{w} = 0 \)

\( \ast \ast \) becomes \( \partial^2 k_z^2 \hat{u} = \frac{f}{k_z} N \hat{\n} (k \times \hat{z}) \)

or \( k_z f \begin{bmatrix}
\hat{u} \\
\hat{v} \\
\hat{w}
\end{bmatrix} = N \hat{\n} \begin{bmatrix}
k_y \\
k_x \\
0
\end{bmatrix} \)

and we can see that

\( \hat{u} = k_y \hat{y}, \hat{v} = -k_x \hat{y}, \hat{w} = \frac{k_z f}{N} \hat{z} \)

where \( \hat{\n} \) is a streamfunction
For \( \hat{\eta} = \frac{N}{|K||\sigma|^2} \rightarrow \)

\[
\hat{\phi}^0 = \begin{bmatrix}
\hat{\psi}^1 \\
\hat{\psi}^2 \\
\hat{\psi}^3 \\
\hat{\psi}^4
\end{bmatrix} = \begin{bmatrix}
1 \\
|K||\sigma|^2 \\
-k_N \\
k_N
\end{bmatrix} \begin{bmatrix}
k y N \\
-k_x N \\
0 \\
k_z F
\end{bmatrix}
\]

For \( k_h = 0 \) \( \hat{\phi}^0 = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \)

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**True Statements**

* The eigenmodes are complete

\[
\hat{\phi}^0 \hat{\phi}^0 = \hat{\phi}^t \hat{\phi}^t = 1 \quad \text{(orthonormal)}
\]

\[
\hat{\phi}^0 \hat{\phi}^t = \hat{\phi}^t \hat{\phi}^0 = 0 \quad \text{(orthogonal)}
\]

\[
k \cdot \hat{\phi}^0 = k \cdot \hat{\phi}^{+*} = 0 \quad \text{(incompressible)}
\]
Since the eigenmodes are complete, we can use them as a basis for the full nonlinear problem!

**Full Nonlinear (inviscid) problem**

\[
\begin{bmatrix}
  u(x,t)
  \\
  \theta(x,t)
\end{bmatrix} = \sum_k \begin{bmatrix}
  \hat{u}(k,t) \\
  \hat{\theta}(k,t)
\end{bmatrix} e^{i k_0 x}
\]

\[
\begin{aligned}
\hat{u}(k,t) &= \sum b_{sk}(k,t) \phi^k(k) e^{-i \sigma_{sk}(k) t} \\
\hat{\theta}(k,t) &= s_k
\end{aligned}
\]

\(s_k = 0, 1, -1\)

\(\sigma_0 = 0 \quad \sigma_{\pm}(k) = \pm \frac{\left(N^2 k_0^2 + k^2 k_z^2\right)^{1/2}}{|k|}\)

Also

\[
p(x,t) = \sum \hat{p}(k,t) e^{i k_0 x}
\]

Plug in and use orthogonality multiple times!
\[
\frac{\partial}{\partial t} b_{mk} = \sum \sum \sum \sum \sum \sum C_{k,p,q} b_{mp} b_{mq} e^{i [\sigma_{mk} + \sigma_{mp} + \sigma_{mq}] t}
\]

with reality condition \( \hat{u}(k) = \hat{u}^*(-k) \)

\# \( k + p + q = 0 \) comes from orthogonality of Fourier modes

\# \( C_{k,p,q} \) is known from the eigenmodes

\# Without waves, we have just written the nonlinear equations in terms of Fourier modes

\# Always have \( k + p + q = 0 \) in Fourier space
* With waves, some special features

* Upon long-time average, the dispersion will tend to significantly reduce the nonlinear effects (time-average of an oscillatory term)

* Recall: \( \sigma^\pm = \pm \left( k_x^2 f^2 + k_y^2 N^2 \right)^{1/2} \)

\[ \sigma^0 = 0 \]

and so the nonlinear effects will tend to zero on average for \( N, f \to \infty \)

\[ (R_o = \frac{u}{L_f}, \quad Fr = \frac{u}{LN} \to 0) \]

* Except for special interactions with

\[ \sigma_{mx} + \sigma_{mp} + \sigma_{mq} = 0 \]

called exact resonances

* Note that interactions between the \( \nu_{12,11}^{ker} \) vertical modes
$\Phi^0(t), \sigma^0 = 0$ are always resonant!

* In a perturbation expansion in powers of \( R_0 = Fr = \varepsilon \rightarrow 0 \), the exact resonances appear at \( O(1) \)

* At next order, there are near-resonances with \( \delta_{mk} + \delta_{mp} + \delta_{mg} = O(\varepsilon) \)

* Thus there is a natural perturbation expansion (weak turbulence theory) for dispersive wave systems

* No such perturbation approach exists for the inviscid NS case without dispersive waves!