Math 322 Chapter 8 | 28

Non-homogeneous boundary conditions & non-homogeneous equations (source terms)

Simple cases

Strategy: try to "remove" the non-homogeneous boundary conditions terms by using a reference solution & then construct the full solution

Example 1

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0$$

$$u(0,t) = A \qquad u(L,t) = B \qquad u(x,0) = f(x)$$

Dirichlet boundary conditions so we expect a steady-state solution, but they are non-homogeneous boundary conditions

We already found the steady-state solution in Chapter 1:

$$u = u_E(x) \quad \text{satisfies} \quad \frac{d^2 u_E(x)}{dx^2} = 0$$

$$\text{with} \quad u_E(0) = A, \quad u_E(L) = B$$

$$\text{gives} \quad u_E(x) = A + \frac{(B-A)}{L} x$$

check

Now define $v(x,t) = u(x,t) - u_E(x)$

to "remove" the non-homogeneous boundary condition from the system for $v(x,t)$

Since $u(x,t) = v(x,t) + u_E(x)$, plug in:

$$\frac{\partial}{\partial t}(v + u_E) = k\frac{\partial^2}{\partial x^2}(v + u_E) \qquad 0 < x < L \quad t > 0$$

$$v(x,t) + u_E(x)\Big|_{x=0} = A \qquad v(x,t) + u_E(x)\Big|_{x=L} = B$$

$$u(x,0) = v(x,0) + u_E(x) = f(x)$$

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} \qquad 0 < x < L \qquad t > 0$$

$$v(0,t) = 0 \qquad v(L,t) = 0 \qquad v(x,0) = f(x) - u_E(x)$$

Back to homogeneous eqn. and homogeneous bcs for $v(x,t)$

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \exp\left[-k\frac{n^2\pi^2}{L^2}t\right]$$

$$f(x) - u_E(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

$$\boxed{\begin{aligned} a_n &= \frac{2}{L}\int_0^L \left(f(x) - u_E(x)\right) \sin\frac{n\pi x}{L}\, dx \\ \text{So} \quad u(x,t) &= u_E(x) + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \exp\left[-k\frac{n^2\pi^2}{L^2}t\right] \\ u_E(x) &= A + \frac{(B-A)}{L}x \end{aligned}}$$

If we had non-zero Newmann bundery conditions and no source terms, this would still work.

If we have non-zero Newmann bundery conditions and a constant source term, this approach would not work because there is no equilibrium solution

Exercise 1.4.6, 8.2.1

$\boxed{\text{Example 2}}$ Using a "reference temp." instead of an "equilibrium temp."

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad 0<x<L\ ,\ t>0$$

$$u(0,t) = A(t) \qquad u(L,t) = B(t) \qquad u(x,0) = f(x)$$

Introduce any (simple) reference temp. satisfying the time-dependent non-homogeneous b.c.s, e.g.

$$r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t)\right]$$

Again let $\quad v(x,t) = u(x,t) - r(x,t)$

$\qquad\qquad\quad u(x,t) = v(x,t) + r(x,t)$ plug in

$$\frac{\partial}{\partial t}(v+r) = k\frac{\partial^2}{\partial x^2}(v+r) + Q \qquad 0<x<L \quad t>0$$

$$u(0,t) = v(0,t) + r(0,t) = A(t) \Rightarrow v(0,t) = 0$$

$$u(L,t) = v(L,t) + r(L,t) = B(t) \Rightarrow v(L,t) = 0$$

$$u(x,0) = f(x) = v(x,0) + r(x,0)$$

$$= v(x,0) + A(0) + \frac{x}{L}\left[B(0) - A(0)\right]$$

The problem for $v(x,t)$ becomes

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \underbrace{Q - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}}_{\bar{Q}} \qquad \begin{array}{c} 0 < x < L \\ t > 0 \end{array}$$

$$v(0,t) = 0 \qquad v(L,t) = 0 \qquad v(x,0) = f(x) - r(x,0)$$

$$= f(x) - A(0) - \frac{x}{L}\left[B(0) - A(0)\right]$$

$$= g(x)$$

In this case $\bar{Q} = Q - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}$

$$= Q - \left(\frac{\partial A}{\partial t} + \frac{x}{L}\left[\frac{\partial B}{\partial t} - \frac{\partial A}{\partial t}\right]\right) + k \cdot 0$$

Recast $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t) \qquad 0 < x < L \quad t > 0$

$v = v(x,t) \qquad v(0,t) = v(L,t) = 0 \qquad v(x,0) = g(x)$

Now we use an eigenfunction expansion

$$v(x,t) = \sum_{\lambda} b_\lambda(t)\phi_\lambda(x)$$

In this case $v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}$

Plug in ; use orthogonality $\Longrightarrow$

$$b_n'(t) + k\frac{n^2\pi^2}{L^2} b_n(t) = \bar{q}_n(t)$$

where $\bar{Q}(x,t) = \sum_{n=1}^{\infty} \bar{q}_n(t) \sin \frac{n\pi x}{L}$

$$\bar{q}_n(t) = \frac{2}{L}\int_0^L \bar{Q}(x,t) \sin \frac{n\pi x}{L}\, dx$$

By the integrating factor method:

$$b_n(t) = b_n(0) \exp\left[-k\frac{n^2\pi^2}{L^2} t\right]$$

$$+ \exp\left[-k\frac{n^2\pi^2}{L^2} t\right] \int_0^t \bar{q}_n(s) \exp\left[k\frac{n^2\pi^2}{L^2} s\right] ds$$

with $v(x,0) = g(x) = \sum_{n=1}^{\infty} b_n(0) \sin\frac{n\pi x}{L}$

$$\Longrightarrow \quad b_n(0) = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

$\boxed{\text{Summary}}$ Example 2

$$u(x,t) = v(x,t) + r(x,t)$$

$$r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t)\right]$$

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$$

$$b_n(t) = b_n(0)\exp\left[-\frac{kn^2\pi^2}{L^2}t\right]$$

$$+ \exp\left[-\frac{kn^2\pi^2}{L^2}t\right]\int_0^t \bar{q}_n(s)\exp\left[\frac{kn^2\pi^2}{L^2}s\right]ds$$

$$\bar{q}_n(t) = \frac{2}{L}\int_0^L \bar{Q}(x,t)\sin\frac{n\pi x}{L}\,dx$$

$$\bar{Q}(x,t) = Q(x,t) - \frac{\partial}{\partial t}r(x,t) + k\frac{\partial^2}{\partial x^2}r(x,t)$$

$$b_n(0) = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$g(x) = f(x) - A(0) - \frac{x}{L}\left[B(0) - A(0)\right]$$