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$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad 0<x<L, \quad t>0$$

$$u(0,t) = A(t) \quad u(L,t) = B(t) \quad u(x,0) = f(x)$$

Let the reference function $r(x,t)$ satisfy the non-homogeneous boundary condition. A simple choice is

$$r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t)\right]$$

Then $u(x,t) = v(x,t) + r(x,t)$; plug in

$$\Longrightarrow \quad \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) - \frac{\partial r(x,t)}{\partial t}$$

$$\boxed{\begin{array}{l} v(0,t) = 0 \qquad v(L,t) = 0 \\ v(x,0) = f(x) - r(x,0) = g(x) \\ \dfrac{\partial v}{\partial t} = k\dfrac{\partial^2 v}{\partial x^2} + \bar{Q}(x,t) \end{array}}$$

$\bar{Q}(x,t)$ known ; $g(x)$ known

$\boxed{\text{Now this is the exam 2 problem :}}$

Let $v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$; plug in

$$\sum_{n=1}^{\infty} b_n'(t) \sin\frac{n\pi x}{L} = -\sum_{n=1}^{\infty} b_n(t)\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}$$

$$+ \sum_{n=1}^{\infty} \bar{q}_n(t) \sin\frac{n\pi x}{L}$$

* We can differentiate term-by-term wrt x

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$$

because $v(x,t)$ satisfies the zero boundary conditions $v(0,t) = v(L,t) = 0$

* We can express

$$\bar{Q}(x,t) = \sum_{n=1}^{\infty} \bar{q}_n(t) \sin\frac{n\pi x}{L}$$

$\{$ even if $\bar{Q}$ does not satisfy $\bar{Q}(0,t) = \bar{Q}(L,t) = 0$ $\}$

Note that

$\bar{q}_n(t)$ is known:

$$\bar{q}_n(t) = \frac{2}{L}\int_0^L \bar{Q}(x,t)\sin\frac{n\pi x}{L}\,dx$$

Orthogonality

$$b_n'(t) + \left(\frac{n\pi}{L}\right)^2 k\, b_n(t) = \bar{q}_n(t)$$

$$\mu(t) = \exp\left[\int \left(\frac{n\pi}{L}\right)^2 k\, dt\right] = \exp\left[\left(\frac{n\pi}{L}\right)^2 k t\right]$$

$$\frac{d}{dt}\left[b_n(t)\,\mu(t)\right] = \bar{q}_n(t)\,\mu(t)$$

$$b_n(t)\,\mu(t) = \int_0^t \mu(s)\,\bar{q}_n(s)\,ds + C$$

$$b_n(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)}\int_0^t \mu(s)\,\bar{q}_n(s)\,ds$$

$$b_n(0) = C$$

$$b_n(t) = \frac{b_n(0)}{\mu(t)} + \frac{1}{\mu(t)}\int_0^t \mu(s)\,\bar{q}_n(s)\,ds$$

$$v(x,0) = g(x) = \sum_{n=1}^{\infty} b_n(0)\sin\frac{n\pi x}{L}$$

$$\Rightarrow b_n(0) = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$\boxed{\text{Summary}}$ Example 2

$$u(x,t) = v(x,t) + r(x,t)$$

$$r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t) \right]$$

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$$

$$b_n(t) = b_n(0) \exp\left[\frac{-kn^2\pi^2}{L^2} t \right]$$

$$+ \exp\left[\frac{-kn^2\pi^2}{L^2} t \right] \int_0^t \bar{q}_n(s) \exp\left[\frac{kn^2\pi^2}{L^2} s \right] ds$$

$$\bar{q}_n(t) = \frac{2}{L} \int_0^L \bar{Q}(x,t) \sin\frac{n\pi x}{L}\, dx$$

$$\bar{Q}(x,t) = Q(x,t) - \frac{\partial}{\partial t} r(x,t) + k\frac{\partial^2}{\partial x^2} r(x,t)$$

$$b_n(0) = \frac{2}{L} \int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

$$g(x) = f(x) - A(0) - \frac{x}{L}\left[B(0) - A(0) \right]$$

What if we have nonhomogeneous boundary conditions on the derivative, or mixed?

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

$$u(x,0) = f(x)$$

e.g. #1 $\quad \frac{\partial u}{\partial x}(0,t) = A(t) \qquad \frac{\partial u}{\partial x}(L,t) = B(t)$

Construct $r(x,t) = \ldots$

$$\frac{\partial r}{\partial x}(0,t) = A(t) \qquad \frac{\partial r}{\partial x}(L,t) = B(t)$$

Then proceed as before with

$$v(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos\frac{n\pi x}{L}$$

* Always construct a reference function satisfying the non-homogeneous boundary conditions

* $\quad v(x,t) = \sum_{\lambda} a_\lambda(t)\,\phi_\lambda(x)$

these change!

Revisit $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$

$u(0,t) = A(t)$ $u(L,t) = B(t)$ $u(x,0) = f(x)$

What if we try

$$u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x) \qquad \left\{ b_n(t) = \frac{\int_0^L u\,\phi_n\,dx}{\int_0^L \phi_n^2\,dx} \right\}$$

where $\phi_n(x)$ satisfies the spatial problem with homogeneous b.c.s, in this case

$$\frac{d^2\phi_n}{dx^2} = -\lambda\phi_n \qquad \phi_n(0) = \phi_n(L) = 0$$

$$\Rightarrow \sin\frac{n\pi x}{L} = \phi_n(x)$$

Now $u(x,t)$ does not satisfy the same (zero) b.c.s as $\phi_n(x)$ and so we cannot differentiate the series term by term.

Lets try to solve without spatial differentiation

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \phi_n(x) = k \frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

Use orthogonality:

$$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \int_0^L \phi_m(x) \phi_n(x) dx$$

$$= \int_0^L \left[k \frac{\partial^2 u}{\partial x^2} + Q(x,t) \right] \phi_m(x) dx$$

$n = m \Rightarrow$

$$\frac{db_m(t)}{dt} = \frac{\int_0^L \left[k \frac{\partial^2 u}{\partial x^2} + Q(x,t) \right] \phi_m(x) dx}{\int_0^L \phi_m^2(x) dx}$$

For $Q(x,t) \sim \sum_{n=1}^{\infty} q_n(t) \phi_n(x)$

$$\int_0^L Q(x,t) \phi_m(x) dx = \int_0^L \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \phi_m(x) dx$$

$n = m \Rightarrow$

$$\int_0^L \varphi(x,t)\,\phi_m(x)\,dx = q_m(t)\int_0^L \phi_m^2(x)\,dx$$

$$\Rightarrow \frac{db_m(t)}{dt} = \frac{\int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_m(x)\,dx + q_m(t)\int_0^L \phi_m^2(x)\,dx}{\int_0^L \phi_m^2(x)\,dx}$$

$$= \frac{\int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_m(x)\,dx}{\int_0^L \phi_m^2(x)\,dx} + q_m(t)$$

integrate by parts twice :

$$\int_0^L \phi_n(x)\frac{\partial^2 u}{\partial x^2}\,dx = \int_0^L u\,\frac{\partial^2 \phi_n(x)}{\partial x^2}\,dx$$

$$- \left(u\frac{\partial \phi_n}{\partial x} - \phi_n\frac{\partial u}{\partial x}\right)\Bigg|_0^L$$

Now use $\phi_n(L) = \phi_n(0) = 0$

$$\phi_n(x) = \sin\frac{n\pi x}{L} \qquad \frac{\partial \phi_n}{\partial x} = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$$

$$\int_0^L \phi_n(x)\frac{\partial^2 u}{\partial x^2}\,dx = -\lambda_n\int_0^L u\,\phi_n\,dx$$

$$- u\,\frac{n\pi}{L}\cos\frac{n\pi x}{L}\Big|_0^L$$

$$= -\lambda_n\int_0^L u\,\phi_n\,dx - \frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]$$

So we have

$$\frac{db_n(t)}{dt} = q_n(t) - K\frac{\left\{\lambda_n\int_0^L u\,\phi_n\,dx + \frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]\right\}}{\int_0^L \phi_n^2(x)\,dx}$$

$$= q_n(t) - K\lambda_n b_n(t)$$

$$- \frac{K\frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]}{\int_0^L \phi_n^2(x)\,dx}$$

$$\frac{db_n(t)}{dt} + k\lambda_n b_n(t) = q_n(t) \frac{-k\frac{n\pi}{L}\left[B(t)(-1)^n - A(t)\right]}{\int_0^L \phi_n^2(x)\,dx}$$

$$= \tilde{q}_n(t)$$

Now proceed as before to solve for $b_n(t)$:

$$b_n(t) = \frac{b_n(0)}{\mu(t)} + \frac{1}{\mu(t)}\int_0^t \tilde{q}_n(s)\,\mu(s)\,ds$$

$$b_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\mu(t) = \exp\left[\left(\frac{n\pi}{L}\right)^2 k t\right]$$

* extra term $= 0$ if homogeneous boundary conditions

* slow convergence otherwise for

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$$ {faster convergence for $v(x,t)$ is an advantage of the reference function method}