Solution of a single, first-order ODE:

* What can we say in general
* What do we know about special equations

Start with the former to dispel the notion that the subject consists of rules that need to be memorized for different classes of equations.

For any 1st-order ODE, we can always try

(i) a graphical solution to obtain at least a qualitative picture

(ii) a numerical approximation, e.g., using finite differences

Today: Graphical Solution for a qualitative picture: Existence & Uniqueness Thm
For IVPs.
Graphical Solution of \[ \frac{dy}{dx} = F(x, y) \quad y(x_0) = y_0 \]

This is IVP = ODE + initial condition

The Method of Direction Fields

The Direction Field is the graph of the direction of the slope \( F(x, y) \) at every point in the \((x, y)\) plane

Example 1

\[ y'(x) = -y(x) \cos x \quad y(\pi) = 2 \]

Linear or Nonlinear? Why?

\((x, y) = (n\pi, y) \quad n \text{ integer } \quad F(x, y) = 0\)

\((x, y) = (2n\pi, y) \quad n \text{ integer } \quad F(x, y) = -y\)

\((x, y) = (3n\pi, y) \quad n \text{ integer } \quad F(x, y) = y\)
* Visual picture of the solution.
* Oscillatory in $x$ and $y$ with period $\pi$.
* $y(x)$ oscillates as $x \to \pm \infty$.

* As $|y|$ increases, the amplitude of oscillation increases but the period remains the same.

* No vertical asymptotes $\Rightarrow$ no singularities $\Rightarrow$ existence for all $x$. 
Existence and Uniqueness

Given \( \frac{dy}{dx} = f(x, y) \), \( y(x_0) = y_0 \), IVP

**If** \( f(x, y) \) **is** real, **continuous** on an open interval containing the initial point \((x_0, y_0)\):

\[ \alpha < x < \beta \quad \text{with} \quad \alpha < x_0 < \beta \]

\[ \gamma < y < \delta \quad \text{with} \quad \gamma < y_0 < \delta \]

Then, at least one solution in some subinterval of \( x \in (\alpha, \beta) \), \( y \in (\gamma, \delta) \).

**If**, in addition, \( \frac{df(x, y)}{dy} \) **is** real and continuous on \( x \in (\alpha, \beta) \), \( y \in (\gamma, \delta) \),

then the solution is unique.
What do you need to do?

Find \((x, y) \in (\delta, \delta)\) where \(F, \frac{\partial F}{\partial y}\) are real
continues containing \(x_0, y_0\).

* Caution: Existence & uniqueness guaranteed in a sub-interval of \((x, y) \in (\delta, \delta)\).

Example 1: \[y' = -y \cos x \quad y(\pi) = 2\]

\[x_0 = \pi, \ y_0 = 2, \ F(x, y) = -y \cos x, \ \frac{\partial F}{\partial y} = -\cos x\]

* \(F(x, y)\) real continuous on \(-\infty < x < \infty, -\infty < y < \infty\)

\[\Rightarrow\] any \((x_0, y_0)\) allowed!

\[\Rightarrow\] at least one solution exists in a sub-interval of \(x \in (-\infty, \infty), y \in (-\infty, \infty)\)

* \(\frac{\partial F}{\partial y}\) real continuous on \(-\infty < x < \infty, -\infty < y < \infty\)

\[\Rightarrow\] the solution is guaranteed unique in a sub-interval of \(x \in (-\infty, \infty), y \in (-\infty, \infty)\)
Example 2 \[ y' = y^{\frac{1}{3}} \quad y(x_0) = 0 \]

Linear or Nonlinear? Why?

Slope Field independent of $x$
Slope $\to 0$ as $y \to 0$

In fact, there are 3 curves corresponding to every point $(x_0, 0)$.

\[ y_{1/2} = \pm \left[ \frac{2}{3} (x-x_0) \right]^{3/2} \quad y_3 = 0 \]

So there is non-uniqueness associated with $y_0 = 0$.
Let's check: \( y_2 = -\sqrt{\frac{2}{3}(x-x_0)} \)

When \( x = x_0 \), then \( y_2 = 0 \) \( \checkmark \)

\[
\begin{align*}
\frac{dy_2}{dx} &= -\frac{2}{3} \left[ \frac{2}{3} (x-x_0) \right]^{1/3} \cdot \frac{3}{2} = - \left[ \frac{2}{3} (x-x_0) \right]^{1/3} \\
y_2^{1/3} &= \left[ -\frac{2}{3} (x-x_0) \right]^{1/3} = - \left[ \frac{2}{3} (x-x_0) \right]^{1/3} \\
\end{align*}
\]

**Thm** of Existence and Uniqueness

\( y' = y^{1/3} \), \( y(x_0) = 0 \)

\( x_0, y_0 = 0 \), \( f(x,y) = y^{1/3} \), \( \frac{\partial F}{\partial y} = \frac{1}{3} y^{-2/3} \)

* \( f(x,y) \) real continuous \(-\infty < x < \infty, \ -\infty < y < \infty \)

\( \Rightarrow \) solution guaranteed to exist in a subrange we can pick that is \( x \neq x_0 \)

* \( \frac{\partial F}{\partial y} \) real continuous in \(-\infty < x < \infty \)

\( 0 < y < \infty, \ -\infty < y < 0 \)

neither contains \( y_0 = 0 \)!
\[ \Rightarrow \text{nonuniqueness for this IVP} \]

\[ \text{Aside: } y' = y^{1/3}, \quad y(x_0) = y_0, \quad y_0 > 0 \]

F, \( \frac{dy}{dx} \) real analytic on \(-\infty < x < \infty, 0 < y < \infty\)

containing in the initial point \((x_0, y_0)\)

\[ \Rightarrow \text{guaranteed a unique solution in} \]

a subrange of \(-\infty < x < \infty, 0 < y < \infty\)

as we can see from the picture.

\[
\begin{bmatrix}
    y = \frac{2}{3} y_0^{3/2} + \frac{2}{3} \left( x-x_0 \right) \\
    \frac{2}{3} \left( x-x_0 \right) > -y_0^{2/3}
\end{bmatrix}
\]
Example 3 \( y' = y^2, \ y(0) = y_0, \ y_0 > 0 \)

Linear or Nonlinear? What is \( x_0? \ y_0? \)

\( f(x,y) = y^2 \) real continuous in \(-\infty < x < \infty, -\infty < y < \infty\)

\( \frac{dF}{dy} = 2y \)

Expect a unique solution in a subrange!

\[ y = \frac{y_0}{1-y_0x} \]

Check!

Where is the solution defined?

\( x > \frac{1}{y_0} \) or \( x < \frac{1}{y_0} \)?

Can \( x \) be both? NO

\(-\infty < x < \frac{1}{y_0}\) because this contains the initial point \( x_0 = 0! \)
Let's make a graph of the solution. For ease of understanding, let's take $y_0 = 2$.

\[ y = \frac{y_0}{1-y_0x} = \frac{2}{1-2x} \]

$x \to -\infty \quad y \to 0^+$
$x \to \frac{1}{y_0} \quad y \to -\infty$

$x \to +\infty \quad y \to 0^-$
$x \to \frac{1}{y_0}^- \quad y \to +\infty$
The solution exists and is unique in \(-\infty < x < \frac{1}{2g_0}\) and we cannot cross over the singularity in solution space.

Example 4.1 \[ y' = \frac{y}{\cos x}, \quad y(\pi /2) = \pi \]

Linear, \(x_0 = \pi /2, \ y_0 = \pi, \ F = \frac{y}{\cos x}, \ \frac{dy}{dx} = \frac{1}{\cos x} \)

\(x_0 = \pi /2\) is a point of discontinuity of \(F\) and \(\frac{dy}{dx} \rightarrow \) the solution is not guaranteed to exist.