Thm of Existence and Uniqueness

For 1st-order ODEs:
also relation to the Direction Fields

Given an Initial Value Problem:

$$\frac{dy}{dx} = F(x, y) ; \text{Initial Condition } y(x_0) = y_0$$

*IF* $F(x, y)$ is real continuous on an open interval containing the initial point $(x_0, y_0)$:

$\alpha \leq x \leq \beta \text{ with } \alpha < x_0 < \beta$

$\gamma \leq y \leq \delta \text{ with } \gamma < y_0 < \delta$

Then there is at least one solution in a subinterval of $x \in (\alpha, \beta)$, $y \in (\gamma, \delta)$ containing $(x_0, y_0)$

*IF* in addition, $\frac{\partial F}{\partial y}$ is real continuous in $x \in (\alpha, \beta)$, $y \in (\gamma, \delta)$

Then the solution is unique.
Note: We are always "starting" at $(x_0, y_0)$.

What do you need to do?: Find $(\alpha, \beta) (\gamma, \delta)$ containing $(x_0, y_0)$ where $F, \frac{\Delta F}{\Delta y}$ are real, continuous existence and uniqueness are guaranteed in a sub-interval of $(\alpha, \beta) (\gamma, \delta)$ containing $(x_0, y_0)$.

Example 1: $y' = -y \cos x$, $y(\pi) = 2$ (linear)

Identify the parts: $x_0 = \pi$, $y_0 = 2$, $F = -y \cos x$, $\frac{\Delta F}{\Delta y} = -\cos x$.

Draw the Direction Field:

Check the Thm:

$F(x, y) = -y \cos x$ is real, continuous on $-\infty < x < \infty$

$-\infty < y < \infty$ (contains $x_0 = \pi$, $y_0 = 2$)

$\Rightarrow$ existence guaranteed in a sub-interval of $-\infty < y < \infty$. 

$\Rightarrow$ start here
* \( \frac{dy}{dx} = -\cos x \) is real, continuous on 
\(-\infty < x < \infty, \quad -\infty < y < \infty \Rightarrow \) soln. is unique

* the picture and the thin are consistent!

Example 2 \( y' = y^{\frac{1}{3}} \quad y(x_0) = 0 \) (Nonlinear)

Identify the Parts: \( x_0 \) a parameter, \( y_0 = 0 \)
\( F(x,y) = y^{\frac{1}{3}} \), \( \frac{dF(x,y)}{dy} = \frac{1}{3} y^{-\frac{2}{3}} \)

Draw the Direction Field

Draw the Direction Field

Check the Thin

* \( F(x,y) \) real continuous on \(-\infty < x < \infty, -\infty < y < \infty \)

\( \Rightarrow \) soln. guaranteed to exist in a subrange
\[ \frac{df(x,y)}{dy} = \frac{1}{3} y^{-2/3} \] is real continuous on \(-\infty < x < \infty\) \[ 0 < y < \infty \] which range \(-\infty < y < 0\) is relevant?

Which range contains the initial point \((x_0, 0)\)?

Neither. We cannot find an open interval containing the initial point where \(\frac{df}{dy}\) is real continuous.

\[ \Rightarrow \] uniqueness is not guaranteed

* the picture and the theorem are consistent!

It is straightforward to check that there are 3 solution curves through \((x_0, 0)\):

\[ y_{1,2} = \pm \left[ \frac{2}{3} (x-x_0) \right]^{3/2}, \quad y_3 = 0 \]
Let's check $y_2 = -\left[ \frac{2}{3} (x-x_0) \right]^{3/2}$

* when $x = x_0$, then $y_2 = 0 \quad \checkmark$

* \[ \frac{dy_2(x)}{dx} = -\frac{3}{2} \left[ \frac{2}{3} (x-x_0) \right]^{1/2} \left( \frac{2}{3} \right) \left[ (x-x_0) \right]^{-1/2} = -\left[ \frac{2}{3} (x-x_0) \right]^{1/2} \]

\[ y_2 = \left\{ \begin{aligned} &-\left[ \frac{2}{3} (x-x_0) \right]^{3/2} \left[ (x-x_0) \right]^{1/2} = -\left[ \frac{2}{3} (x-x_0) \right]^{1/2} \\ \end{aligned} \right. \]

\[ \frac{dy_2(x)}{dx} = y_2^{1/3} \quad \checkmark \]

* What if we change the initial point to $(x_0, y_0)$ with $y_0 > 0$

Looks like we could follow a unique curve "near" $(x_0, y_0)$
What does the Thm say?

\[ y' = y^{1/3}, \quad y(x_0) = y_0 > 0 \]

* \( F(x, y) = y^{1/3} \) real continuous on \( -\infty < y < \infty \)
  
  \[ \Rightarrow \text{existence guaranteed} \]

* \( \frac{dF}{dy} = \frac{1}{3} y^{-2/3} \) real continuous on

\[ -\infty < x < \infty \quad 0 < y < \infty \] \( \bigcup \quad -\infty < y < 0 \]

Which is relevant?

Since \( y_0 > 0 \), the range \( 0 < y < \infty \) contains
the initial point.

\[ \Rightarrow \text{uniqueness guaranteed in a} \]

subrange of \( -\infty < x < \infty \), \( 0 < y < \infty \)

* consistent with the picture
Challenge: Is this a solution to

\[ y' = y^{\frac{1}{3}}, \quad y(x_0) = y_0, \quad y_0 > 0 \]

\[ y = \sqrt[3]{y_0} + \frac{2}{3} (x-x_0)^{\frac{3}{2}} \]

For what values of \( x \) is there a unique solution?

For what values of \( y \) is there a unique solution?
Example 3: \( y' = y^2 \), \( y(0) = y_0 > 0 \)

Linear or Nonlinear? What is \( x_0, y_0 \)?
\( f(x,y) \)? \( \frac{df(x,y)}{dy} \)?

\( f(x,y) = y^2 \) real continuous on \( -\infty < x < \infty, -\infty < y < \infty \)
\( \frac{df(x,y)}{dy} = 2y \)

Expect a unique soln. near \((0,y_0)\) in a
subrange of \( -\infty < x < \infty \), \( -\infty < y < \infty \)

Recall the
Direction Field

By Separation, one can find the solution
\[ y(x) = \frac{y_0}{(1-y_0x)} \quad \square \text{Check} \]
Where is the solution defined?

\[ x > \frac{1}{y_0} \quad \text{or} \quad x < \frac{1}{y_0} \quad ? \]

Can it be both? No! Why not?

The soln. is defined in \(-\infty < x < \frac{1}{y_0}\) because this interval contains \((0, y_0)\) as can be seen from the plot.

Make a graph of the solution, e.g., for \(y_0 = 2\):

\[ y = \frac{2}{1 - 2x} \] \; \text{vertical asymptote at} \; x = \frac{1}{y_0} = \frac{1}{2}

Looks like the Direction Field!
\[ x \to -\infty, \quad y \to 0^+ \]

The solution exists and is unique in \(-\infty < x \leq \frac{1}{y_0}\).

\[ x \to +\infty, \quad y \to 0^- \]

\[ x \to \frac{1}{y_0}, \quad y \to \infty \]

\[ x \to \frac{1}{y_0}, \quad y \to -\infty \]

**Example 4**

\[ y' = \frac{y}{\cos x}, \quad y\left(\frac{\pi}{2}\right) = \pi \]

Linear, \(x_0 = \frac{\pi}{2}, \quad y_0 = \pi, \quad F = \frac{y}{\cos x}, \quad \frac{\partial F}{\partial y} = \frac{1}{\cos x}, \quad \frac{\partial F}{\partial x} = \frac{y}{\cos^2 x} \]

\(x_0 = \frac{\pi}{2}\) is a point of discontinuity of \(F(x,y)\) and \(\frac{\partial F}{\partial y}(x,y) \Rightarrow \) the solution is not guaranteed to exist.