Math 322 Lecture 3 Sections 1.3, 1.4

Goals

1. What kind of boundary conditions make sense for the 1D heat eqn.

2. Notion of a steady state solution; long times when $\frac{\partial T}{\partial t} = 0$. Does it always exist?

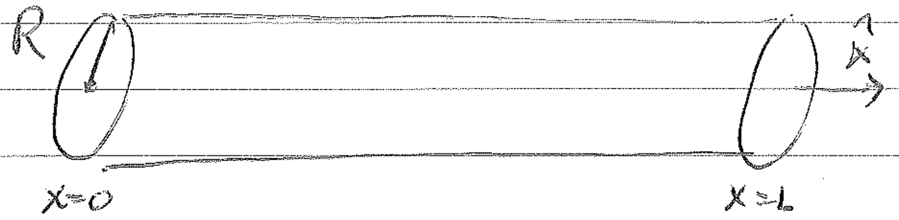
3. Show that changing the b.c.s can fundamentally change the nature of the steady-state soln.

* Prescribed, fixed T at both ends $\Rightarrow$ a unique steady-state soln. independent of initial conditions

* Perfectly insulated b.c.s (both ends) $\Rightarrow$ a non-unique steady-state solution until the initial condition is specified

Sections 1.3, 1.4

Math 322 Lecture 3 | 1D Heat Eqn.



$L >> 1$
$R << 1$

$\Longrightarrow$

$$\frac{\partial T(x,t)}{\partial t} = k_0 \frac{\partial^2 T(x,t)}{\partial x^2} \quad \text{(no sources / sinks in } 0 < x < L\text{)}$$

What kind of boundary conditions make sense?

1. Fixed T (Prescribed, Dirichlet)

e.g. $T(0,t) = T_1(t) \qquad T(L,t) = T_2(t)$

2. Fixed Flux $\{ \underline{q} = -k_0 \underline{\nabla} T \}$

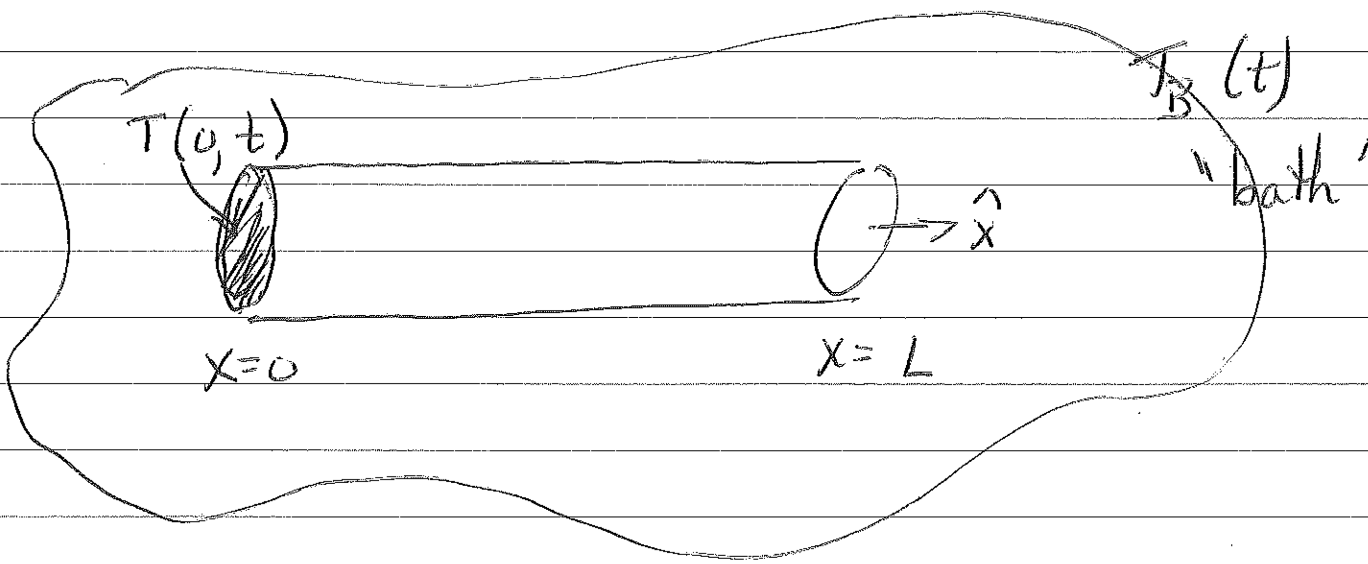
e.g. $-k_0 \frac{\partial T}{\partial x}(0,t) = \phi(t)$

or $\frac{\partial T}{\partial x}(0,t) = \phi^*(t)$ (Neumann)

"Perfectly insulated" or "adiabatic" $\Longrightarrow$

$\frac{\partial T}{\partial x}(0,t) = 0$ no heat flux through the boundary at $x=0$.

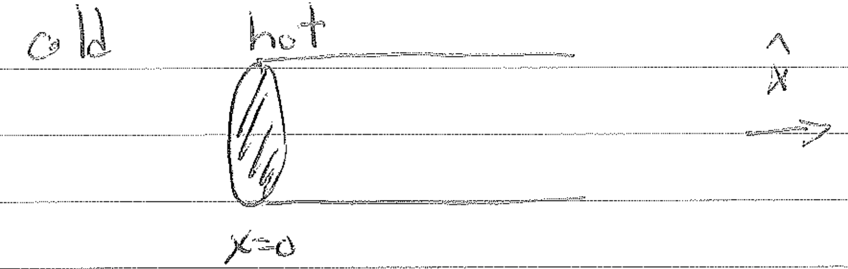
3. Newton's Law of Cooling: heat flux at the boundary is proportional to the difference between T at the boundary and T in the "bath."



$$-k_0 \frac{\partial T}{\partial x}(0,t) = -H\left(T(0,t) - T_B(t)\right)$$

$\uparrow$ proportionality constant

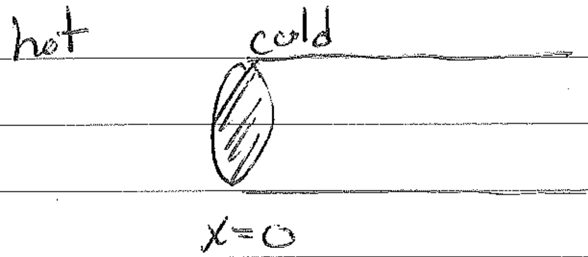
Assume $k_0 > 0$, $H > 0$. Is the sign correct?



$$\frac{\partial T}{\partial x}(0,t) = \frac{H}{K_0}\left(T(0,t) - T_B(t)\right)$$

$$>0 \qquad (+) \qquad >0$$

✓



$$\frac{\partial T}{\partial x}(0,t) = \frac{H}{K_0}\left(T(0,t) - T_B(t)\right)$$

$$<0 \qquad (+) \qquad <0$$

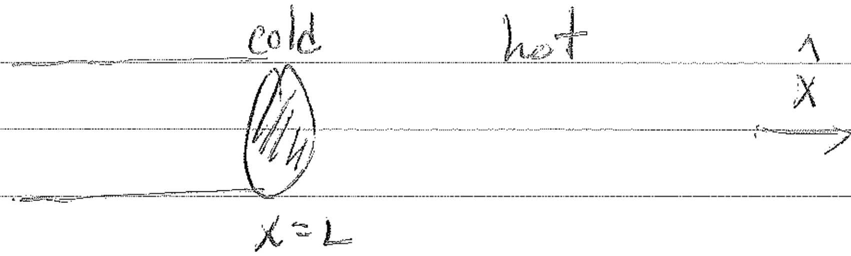
✓

What formula should we use at $x = L$

$$-K_0 \frac{\partial T}{\partial x}(L,t) = H\left(T(L,t) - T_B(t)\right)$$

Check

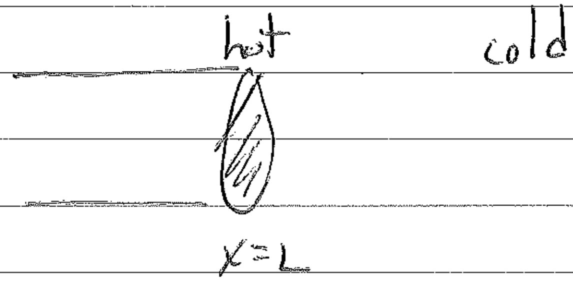
(3.5)



$$\frac{\partial T}{\partial x}(L,t) = -\frac{H}{K_0}\left(T(L,t) - T_B(t)\right)$$

>0 $(-)$ <0

$+$ $(-)$ $(-)$ ✓



$$\frac{\partial T}{\partial x}(L,t) = \frac{-H}{K_0}\left(T(L,t) - T_B(t)\right)$$

<0 $(-)$ >0

$(-)$ $(-)$ $(+)$ ✓

For the 1D problem, we need one of these boundary conditions at each endpoint and an initial condition

Example # 1 $\quad \frac{\partial T}{\partial t} = k\,\frac{\partial^2 T}{\partial x^2} \qquad 0 < x < L$

$T(x,0) = f(x)\,,\quad T(0,t) = T_1(t)\,,\quad T(L,t) = T_2(t)$

Example # 2 $\quad \frac{\partial T}{\partial t} = k\,\frac{\partial^2 T}{\partial x^2} \qquad 0 < x < L$

$T(x,0) = f(x)\,,\quad T(0,t) = T_1\,,\quad T(L,t) = T_2$

T_1 constant $\qquad T_2$ constant

Is there a "steady state" solution with

$\frac{\partial T}{\partial t} = 0$? Expect "no" for #1 ;

"yes" for # 2

$\boxed{\text{Example \#2}}$ Lets look for a steady state soln.

$T = T_S(x)$ with $k \dfrac{d^2 T_S(x)}{dx^2} = 0 \quad 0 < x < L$

(total derivatives) $T_S(0) = T_1 \quad T_S(L) = T_2$

$\Rightarrow \quad T_S(x) = C_1 x + C_2$ (from eqn.)

$T_S(0) = C_2 = T_1$ (from b.c.)

$T_S(L) = C_1 L + C_2 = T_2$ (from b.c.)

or $C_1 = (T_2 - T_1) / L$

$\Rightarrow \quad T_S(x) = \dfrac{(T_2 - T_1)}{L} x + T_1$ a

linear profile ; no information about the initial condition is necessary!

{ Approach to steady state in 8.2 }

Perfectly Insulated

Lets change the boundary conditions:

$\boxed{\text{Example Problem \#3}}$ $\quad \dfrac{\partial T}{\partial t} = K \dfrac{\partial^2 T}{\partial x^2} \qquad T(x,t)$

$$T(x,0) = f(x), \quad \frac{\partial T}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(L,t) = 0$$

Lets look for a steady state solution with $\dfrac{\partial T}{\partial t} = 0$.

$$K \frac{d^2 T}{dx^2} = 0 \quad \Rightarrow \quad T_s(x) = C_1 x + C_2$$

$$\frac{\partial T_s}{\partial x}(x) = C_1$$

both conditions $\quad \dfrac{\partial T_s}{\partial x}(0) = \dfrac{\partial T_s}{\partial x}(L) = C_1 = 0$

$\Rightarrow \quad T_s(x) = C_2 \quad$ not uniquely determined by the boundary conditions

We need more information. In this case, since both ends are perfectly insulated, the total thermal energy is conserved:

$$\Rightarrow \quad \frac{d}{dt} \int_0^L c\rho_0 T \, dx = -K_0 \frac{\partial T}{\partial x}(0,t) + K_0 \frac{\partial T}{\partial x}(L,t)$$

$$= 0$$

$$\int_0^L c_0 \rho_0 T \, dx = \text{constant in time}$$

The initial condition tells us what the energy is at $t=0$, and therefore we know the energy for all time

$$t=0 \qquad \int_0^L c_0 \rho_0 f(x) \, dx = A \quad \text{constant}$$

$$t>0 \;\{\text{long time}\} \qquad \int_0^L c_0 \rho_0 C_2 \, dx = A$$

$$\Rightarrow \qquad \int_0^L f(x) \, dx = C_2 L$$

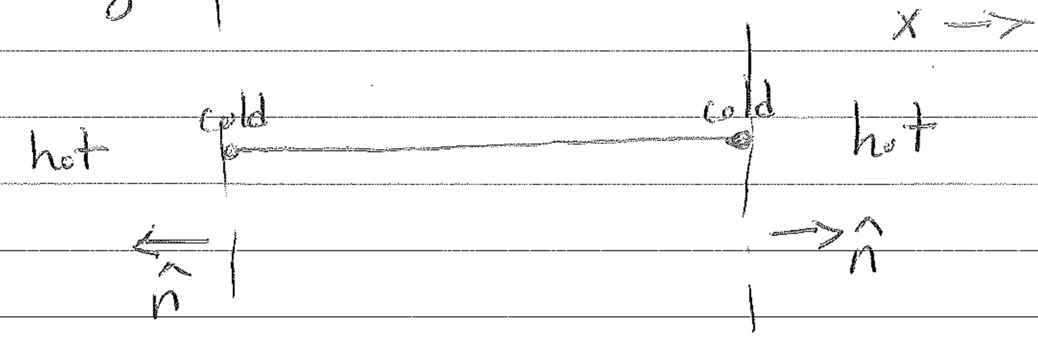
$$\Rightarrow \qquad C_2 = \frac{1}{L}\int_0^L f(x) \, dx$$

the average of the initial temperature distribution

$$T_s(x) = C_2 = \frac{1}{L}\int_0^L f(x) \, dx$$

Check signs again

e.g.



expect heat entering at both ends

$$\Rightarrow \quad -k_0 \underline{\nabla} T \cdot (-\hat{n}) > 0$$

$$\left\{ \underset{\sim}{q} \cdot (-\hat{n}) > 0 \right\}$$

Left $-\hat{n} = \hat{x}$ | Right $-\hat{n} = -\hat{x}$

$$-k_0 \frac{\partial T}{\partial x} \hat{x} \cdot \hat{x} \qquad\qquad -k_0 \frac{\partial T}{\partial x} \hat{x} \cdot (-\hat{x})$$

$$-k_0 \frac{\partial T}{\partial x}(0,t) \qquad\qquad k_0 \frac{\partial T}{\partial x}(L,t)$$