We noticed that simple shear flows (1D-1C) Couette and Plane Channel Flows have straight streamlines but nonzero vorticity.

Streamlines are everywhere tangent to the velocity vector \( \mathbf{u}(x, t) \).

In Cartesian coordinates, a streamline \( x = x(s), y = y(s), z = z(s) \) is obtained from

\[
\frac{dx}{ds} = \frac{dy}{ds} = \frac{dz}{ds}
\]

where

\[
\alpha = (u, v, w) \quad \text{and} \quad s \quad \text{is arc length}.
\]

In steady 2D flow

\[
\tan \theta = \frac{dy}{dx} = \frac{V}{u}
\]

\[
\left\{ \frac{\Delta x}{\Delta s} = \cos \theta, \quad \frac{\Delta y}{\Delta s} = \sin \theta \right\}
\]
A toy model for solid body rotation in 2D: $u(x,y) = -By\hat{x} + Bx\hat{y}$ has circular streamlines: \[
dy = -\frac{x}{y} \partial x \Rightarrow x^2 + y^2 = C\]

Is this an acceptable flow?

Calculate the vorticity $\omega = \nabla \times u = 2B\hat{z}$

But the "irrotational vortex"

$u(r,\theta) = \omega \hat{r} + u_0 \hat{\theta} = \frac{\rho}{2\pi r} \hat{\theta}$

Is this an acceptable flow?
\[ \nabla \times \mathbf{u} = 2 \left[ \frac{1}{r} \frac{1}{r} (ru_\theta) - \frac{1}{r} \frac{du}{d\theta} \right] = 0 \]

Lesson: straight streamlines \( \Rightarrow \) zero vorticity

Circular streamlines \( \Rightarrow \) nonzero vorticity

Because vorticity is local spin

\[
\begin{align*}
\text{viscosity} & \Rightarrow \text{no-slip condition} \Rightarrow \text{shear flow} \\
\text{near solid boundaries} & \Rightarrow \nabla \times \mathbf{u} \neq 0
\end{align*}
\]

Zero vorticity (irrotationality) is thus associated with inviscid flow \((\nu = 0)\).

Note that \(\nu = 0\) is not the same as \(\nu \to 0\)!

**Irrotational Flow** \(\nu = 0\)

\[ \nabla \times \mathbf{u} = 0 \]

\[ \Rightarrow \mathbf{u} = -\nabla \psi \text{ because } \nabla \times (\nabla \psi) = 0 \]

Then if the flow is also incompressible \(\nabla \cdot \mathbf{u} = 0\)

\[ \Rightarrow \nabla \cdot (-\nabla \psi) = 0 \Rightarrow \nabla^2 \psi = 0 \]

So for irrotational, incompressible flow,

one can solve the linear equation
\[ \nabla^2 \phi = 0 \text{ with appropriate boundary conditions.} \]

Then find \( u = -\nabla \phi \)

Instead of solving the nonlinear system \( \left\{ \begin{array}{l} \nabla \cdot u = 0, \\
\frac{du}{dt} + (u \cdot \nabla) u = -\frac{1}{\rho_0} \nabla p 
\end{array} \right\} \)

2D-2C Incompressible Flow (Cartesian)

There is a scalar streamfunction \( \psi(x, y) \)

such that \( u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \)

since \( \nabla \cdot u = 0 = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \)

Then if the flow is also irrotational \( \nabla \times u = 0 \)

\[ \Rightarrow \nabla^2 \psi(x, y) = 0 \text{ with appropriate boundary conditions.} \]
There is a famous equation called Bernoulli's Equation pertaining to inviscid, incompressible flow of a Newtonian fluid:

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho_0} \nabla p + \mathbf{g} \]

Using \( (u \cdot \nabla) u = \frac{1}{2} \nabla (u^2) + \omega \times u \Rightarrow \)

\[ \frac{\partial u}{\partial t} + \omega \times u = \nabla \left( \frac{-u^2}{2} - \frac{P}{\rho_0} + g z \right) \]  

where \( \mathbf{g} \) is the direction of gravity.

Now assume flow along a streamline with line element \( ds \) (\( ds \) along a streamline):

\[ ds \cdot \left[ \frac{\partial u}{\partial t} + \omega \times u = -\nabla \left( \frac{u^2}{2} + \frac{P}{\rho_0} + g z \right) \right] \]

Also since \( ds \parallel u \) and \( \omega \times u \perp u \),

then \( ds \cdot (\omega \times u) = 0 \Rightarrow \)

\[ \frac{\partial u}{\partial t} \cdot ds = -\nabla \left( \frac{u^2}{2} + \frac{P}{\rho_0} + g z \right) \cdot ds \]

with \( ds \) along a streamline.
Integrate:
\[ \int \frac{d\mathbf{u}}{dt} \cdot d\mathbf{r} = -\int \nabla \left( \frac{u^2}{2} + \frac{p}{\rho_o} + g z \right) \cdot d\mathbf{r} \]

\[ = -\left( \frac{u^2}{2} + \frac{p}{\rho_o} + g z \right) + C \]

along a streamline, where \( C \) is a different constant for each streamline; unsteady Bernoulli.

In the special case of irrotational flow with \( \mathbf{w} = \nabla \times \mathbf{u} = 0 \) so that \( \mathbf{w} \times \mathbf{u} = 0 \),

then \( d\mathbf{r} \) can be any line element (not necessarily along a streamline); then

\[ \int \frac{d\mathbf{u}}{dt} \cdot d\mathbf{r} = -\left( \frac{u^2}{2} + \frac{p}{\rho_o} + g z \right) + C^* \]

where \( C^* \) is the same constant everywhere in the flow.
Steady Bernoulli for inviscid, incompressible, irrotational flow:

\[ \frac{u_1^2}{2} + \frac{p_1}{\rho_0} + g z_1 = \frac{u_2^2}{2} + \frac{p_2}{\rho_0} + g z_2 \]

For all points in the flow given a velocity, one can find the pressure.

Come back to this question of: "Is it reasonable to assume irrotational flow?"

\[ \text{Yes if the flow is axisymmetric...} \]

Elementary Airfoil Theory:

1. Assume inviscid, incompressible, irrotational, 2D DC flow

2. Solve \( \nabla^2 Y = 0 \) or \( \nabla^2 Y = 0 + b \cos \theta \)

3. Find \( u \) from \( Y \) or \( Y \)

4. Find \( p \) from Bernoulli

5. Find lift and drag from \( p \)
We've learned that simple 1D-1C, steady shear flows have vorticity associated with viscosity and the no-slip condition at solid boundaries.

Let's see what we can learn about vorticity from a simple, time-dependent shear flow.

Consider semi-infinite flow above a flat plane. At time $t=0$ we impulsively set the plane into motion so that $u = u(y,t) \hat{x}$.

![Diagram of flow](image)

$u(y,0) = 0 \quad y > 0$

$u(0,t) = u_0 \quad t > 0$

$u(\infty, t) = 0 \quad t > 0$

Automatically satisfies $\nabla \cdot u = 0$. 

Math 705
\[ \dot{y}, \dot{z} \text{ momentum} \Rightarrow \frac{df}{dy} = \frac{df}{dz} = 0 \]

\[ \dot{x} \text{ momentum: } \frac{du}{dt} + (u \cdot \nabla) u = -\frac{1}{ho} \nabla p + \nu \nabla^2 u \]
(neglect gravity)

Thus \( p = p(x,t) \), \( u = u(y,t) \) and

\[ -\frac{1}{\rho} \frac{dp(x,t)}{dx} = \frac{d}{dt} \frac{d}{dy^2} u(y,t) = c(t) \]

Then \( p(x,t) = -\rho c(t) x + D \)

but with \( p(\infty) = p(-\infty) \) \( \frac{dp(x,t)}{dx} = 0 \)

\[ \Rightarrow \ p = D = p_\infty \]

\[ \Rightarrow \ \frac{d}{dt} \frac{d}{dy^2} u(y,t) = \nu \frac{d^2}{dy^2} u(y,t) \]

and this is just the heat equation, and we know that there are special similarity solutions

\[ u(y,t) = F \left( \frac{y}{\sqrt{t \nu}} \right) \] where \( \xi = \frac{y}{\sqrt{t \nu}} \)

is a similarity variable
Remind that \( \frac{du}{dt} = \nu \frac{du}{dy} \) is invariant under the change of variables \( y = \alpha x', t = \alpha^2 t' \) which motivates the similarity variable \( \xi = \frac{y}{t^{1/2}} \).

In fact we can look for a solution of the form

\[ u(y, t) = \xi(y), \quad \xi = \frac{y}{(2t)^{1/2}} \]

\[ \frac{du}{dt} = \xi'(y) \frac{dy}{dt} = \xi'(y) \int \left[ -\frac{1}{2} \frac{dy}{(2t)^{1/2}} + \frac{1}{t^{3/2}} \right] \]

\[ \frac{d^2u}{dy^2} = \xi''(y) \frac{1}{2} \]

Plugging in \( \Rightarrow \)

\[ \xi''(y) + \frac{1}{2} \left( \frac{y}{(2t)^{1/2}} \right) \xi'(y) = 0 \]

\[ \xi''(y) + \frac{1}{2} \xi'(y) = 0 \]

which has solution

\[ \xi(y) = A + B \int_0^y \exp \left( -\frac{s^2}{4} \right) ds \]
What happens to the boundary conditions?

\[ u(y,0) = 0 \quad \text{and} \quad u(\infty, t) = 0 \quad \Rightarrow \quad f(\infty) = 0 \]

\[ u(0, t) = u_0 \quad \Rightarrow \quad f(\eta = 0) = u_0 \quad \Rightarrow \quad A = u_0 \]

Then \[ f(\eta = \infty) = 0 \quad \Rightarrow \]

\[ 0 = u_0 + B \int_{\infty}^{\infty} \exp \left( -\frac{s^2}{4} \right) ds = u_0 + B \sqrt{\pi} \]

\[ \Rightarrow \quad f = u_0 \left[ 1 - \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \exp \left( -\frac{s^2}{4} \right) ds \right] \]

\((\text{pictures})\)
\[
\frac{u}{u_0} = 1 - \frac{1}{\sqrt{\pi}} \int_0^t e^{x^2} \frac{e^{-x^2}}{4} dx
\]

\[
\eta = \frac{y}{(y + t)^{1/2}}, \quad \nu = 1
\]
What happens to vorticity?

\[ \omega = \omega_x \]

\[ \omega = -\frac{du}{dy} = -\left( -\frac{u_0}{\sqrt{\pi t}} \right) \exp \left( -\frac{y^2}{4t} \right) \left( \frac{1}{\sqrt{\pi t}} \right)^2 \]

\[ = -\frac{du}{dy} \frac{\partial y}{\partial y} = \frac{u_0}{\sqrt{\pi vt}} \exp \left[ -\frac{1}{4} \frac{y^2}{v t} \right] \]

which is exponentially small when \( \frac{1}{4} \frac{y^2}{v t} = O(1) \), or when \( y = O((vt)^{1/2}) \).

Initially there is a "vortex sheet" at \( y = 0 \); all the vorticity \( \omega = -\frac{du}{dy} \) is concentrated at the boundary \( y = 0 \) (a sheet at \( y = 0 \)).

Then in time, the vorticity spreads a distance \( y = O((vt)^{1/2}) \); vorticity "diffuses" or distance of order \( (vt)^{1/2} \) in time \( t \); the diffusion time is \( t = O(\frac{y^2}{v}) \) or \( t = O(\frac{L^2}{v}) \).
Conclusion: Viscosity acts to diffuse (spread) vorticity in time $t = O \left( \frac{L^2}{\nu} \right)$ over distance $L = O \left( \sqrt{v t} \right)$.

Shear flows have vorticity, and vorticity is associated with walls and viscosity.

Taking the curl of the eqn. for $u$ (no gravity):

$$\frac{d\omega}{dt} + \left( u \cdot \nabla \right) \omega - \left( \omega \cdot \nabla \right) u = \nu \nabla^2 \omega$$

* In the 1D shear flow just discussed, there is diffusion of vorticity generated by viscosity at a wall: $\nu \nabla^2 \omega$

* In 2D, there can also be advection of vorticity by the term $(u \cdot \nabla) \omega$.

* In 3D, there can also be stretching of vorticity by $(\omega \cdot \nabla) u$. 
The stretching term \((\omega \cdot \nabla)u\) is what may lead to possible singularities in 3D. Adding a top wall