Math 322 | 30

Non-homogeneous Equations / Non-homogeneous Boundary Conditions

1D heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad 0<x<L, \; t>0$$

$$u(0,t) = A(t) \quad u(L,t) = B(t) \quad u(x,0) = f(x)$$

Option 1 Let $u(x,t) = v(x,t) + r(x,t)$
where $r(x,t)$ satisfies $r(0,t) = A(t)$ $r(L,t) = B(t)$;
$v(x,t)$ satisfies $v(0,t) = v(L,t) = 0$

Then $v(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$ (fast convergence)

$$b_n'(t) + \left(\frac{n\pi}{L}\right)^2 k\, b_n(t) = \bar{q}_n(t)$$

Option 2 Let $u(x,t) = \sum_{n=1}^{\infty} \tilde{b}_n(t) \sin\frac{n\pi x}{L}$

with slow convergence

Without term-by-term spatial differentiation, we can derive:

$$\tilde{b}_n'(t) + \left(\frac{n\pi}{L}\right)^2 K \tilde{b}_n(t) = \tilde{q}_n(t)$$

$$-K\frac{2}{L}\left(\frac{n\pi}{L}\right)\left[B(t)(-1)^n - A(t)\right] = \tilde{q}_n(t)$$

these terms come from boundary terms following the integration by parts

Option 1

Pros: * differentiation term by term allowed
* fast convergence

Cons: * Need the reference function $r(x,t)$

Option 2

Pros: * no reference function necessary

Cons: * no term by term differentiation allowed; need integration by parts instead
* slow convergence

What if we change the type of non-homogeneous boundary conditions?

e.g. $\frac{\partial u}{\partial x}(0,t) = A(t) \qquad \frac{\partial u}{\partial x}(L,t) = B(t)$

or $\frac{\partial u}{\partial x}(0,t) = A(t) \qquad u(L,t) = B(t)$

etc. ?

$\boxed{\text{Option 1 and Option 2}}$ same as before

with different (appropriate) eigenvalues/eigenfunctions

e.g. for $\frac{\partial u}{\partial x}(0,t) = A(t) \qquad \frac{\partial u}{\partial x}(L,t) = B(t)$

$\boxed{\text{Option 1}}$ $r(x,t) = \ldots$ with

$$\frac{\partial r(0,t)}{\partial x} = A(t) \qquad \frac{\partial r(L,t)}{\partial x} = B(t)$$

$$v(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi x}{L}$$

$\boxed{\text{Option 2}}$ $u(x,t) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}$

More abstractly, we are always using

$$v(x,t) = \sum_{\lambda} a_\lambda(t)\,\phi_\lambda(x) \quad \text{or}$$

$$u(x,t) = \sum_{\lambda} a_\lambda(t)\,\phi_\lambda(x)$$

Now we can generalize $\boxed{\text{Option } 1, 2}$ to higher dimensions / different geometries / different operators

E.g. Vibrating Membranes given by the Non-homogeneous Wave Equation in 2D

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + \varphi(\underline{x},t) \qquad u = u(\underline{x},t)$$

Keep the physically relevant zero-displacement boundary condition

$$u\big|_{\Omega} = 0$$

Initial displacement / velocity :

$$u(\underline{x},0) = \alpha(\underline{x}) \qquad \frac{\partial u}{\partial t}(\underline{x},0) = \beta(\underline{x})$$

Two methods to proceed

① Plug in $u(\underline{x},t) = \sum_i A_i(t)\,\phi_i(\underline{x})$ and differentiate term by term ; ok because of the zero boundary condition.

② Integrate by parts

Here both methods yield the same result because of the zero boundary condition, but method 2 can be use for non-zero b.c.s

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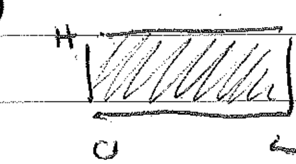
In both cases $\varphi(x,t) = \sum_i q_i(t)\,\phi_i(\underline{x})$

Lets compare Methods 1, 2 for 2D Cartesian ; then



$\sum_i A_i(t)\,\phi_i(\underline{x})$ is shorthand for

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(t)\sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}$$

$\boxed{\text{Method 1}}$

$$\sum_i \frac{d^2 A_i(t)}{dt^2} \phi_i(\underline{x}) = c^2 \sum_i (-\lambda_i) A_i(t) \phi_i(\underline{x})$$

$$+ \sum_i q_i(t) \phi_i(\underline{x})$$

following from $\nabla^2 \phi_i(\underline{x}) = -\lambda_i \phi_i(\underline{x})$

Use orthogonality : $\iint \phi_i \phi_j \, d\underline{x} = \iint \phi_i^2 \delta_{ij} \, d\underline{x}$

$\Longrightarrow$

$$\frac{d^2 A_i(t)}{dt^2} + c^2 \lambda_i A_i(t) = q_i(t) \qquad \boxed{\text{solve}}$$

where $q_i(t) = \dfrac{\iint Q \, \phi_i \, dx}{\iint \phi_i^2 \, dx}$

$$\lambda_i = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

Method 2

$$\sum_i \frac{d^2 A_i(t)}{dt^2} \phi_i(\underline{x}) = c^2 \nabla^2 u(\underline{x}, t) + \sum_i q_i(t) \phi_i(\underline{x})$$

Use orthogonality $\iint \phi_i \phi_j \, d\underline{x} = \iint \phi_i^2 \delta_{ij} \, d\underline{x}$

$\Longrightarrow$

$$\frac{d^2 A_i(t)}{dt^2} = \frac{\iint c^2 \nabla^2 u \, \phi_i \, d\underline{x}}{\iint \phi_i^2 \, d\underline{x}} + q_i(t)$$

Now integrate by parts twice

$$\iint \nabla^2 u \, \phi_i \, d\underline{x} = \iint u \nabla^2 \phi_i \, d\underline{x}$$

$$+ \underbrace{\oint \phi_i \nabla u \cdot \hat{n} \, ds - \oint u \nabla \phi_i \cdot \hat{n} \, ds}_{\text{boundary terms}}$$

here they are zero, but if not we may proceed ...

$$A_i''(t) = \frac{\iint c^2 u \nabla^2 \phi_i \, d\underline{x}}{\iint \phi_i^2 \, d\underline{x}} + q_i(t)$$

$$= \frac{\iint c^2 u (-\lambda_i) \phi_i \, d\underline{x}}{\iint \phi_i^2 \, d\underline{x}} + q_i(t)$$

$$= -\lambda_i c^2 A_i(t) + q_i(t)$$

since $u(\underline{x}, t) = \sum_i A_i(t) \phi_i(\underline{x})$ and

$$A_i(t) = \frac{\iint u \phi_i \, d\underline{x}}{\iint \phi_i^2 \, d\underline{x}}$$

$A_i''(t) + \lambda_i c^2 A_i(t) = q_i(t)$ is the same as for Method 1