\[ \frac{d\rho}{dt} + c(\rho) \frac{\partial \rho}{\partial x} = 0 \quad \rho(x,0) = f(x) \quad \Rightarrow \]

\[ \frac{d\rho}{dt} = 0 \quad \text{along} \quad x(t) = c(\rho_0) t + x_0 \]

\[ \rho(0) = \rho_0 = f(x_0) \]

We want to consider the situation of crossing characteristics.

\[ x = c(\rho_2) t + x_2 \]

\[ x = c(\rho_1) t + x_1 \]

\[ \text{with} \quad c(\rho_1) > c(\rho_2) \]

\[ -u_{\max} \leq c(\rho) \leq u_{\max} \]

same as lower density traffic behind higher density traffic.

faster traffic behind slower traffic
\( p(x,t) \) steepens and eventually becomes multi-valued

\[ \int_{x_s(t)}^{p^+} dx \]

and the shock location is a function of time that we need to find.

Since the PDE is no longer valid when \( p(x,t) \) is multi-valued, go back to the integral formulation of the conservation law:

\[ \frac{d}{dt} \int_{a}^{b} p(x,t) \, dx = q(a,t) - q(b,t) \]

\[ q(x,t) = u(x,t) \rho(x,t) \]
\[
\frac{d}{dt} \int_{x_{S}^{-}(t)}^{x_{S}^{+}(t)} p(x,t) \, dx + \frac{d}{dt} \int_{x_{S}^{+}(t)}^{b} p(x,t) \, dx
\]

\[
= q(a,t) - q(b,t)
\]

\[
x_{S}^{+}(t) = x_{S}(t) + \varepsilon, \quad x_{S}^{-}(t) = x_{S}(t) - \varepsilon
\]

evitably let \( \varepsilon \to 0 \)

By Leibniz's Rule,

\[
\int_{x_{S}^{-}(t)}^{x_{S}^{+}(t)} \frac{df}{dt} \, dx + p(x_{S}^{-}(t)) \frac{dx_{S}^{-}(t)}{dt}
\]

\[
+ \int_{x_{S}^{+}(t)}^{b} \frac{df}{dt} \, dx - p(x_{S}^{+}(t)) \frac{dx_{S}^{+}(t)}{dt} = q(a,t) - q(b,t)
\]

But the PDE is still valid for

\[
[a, x_{S}^{-}(t)] \quad [x_{S}^{+}(t), b]
\]

\[
\Rightarrow
\]
\[ \int_a^{x_s^-(t)} - \frac{dq}{dx} \, dx + \rho(x_s^-, t) \frac{dx_s^-(t)}{dt} \]
\[ + \int_{x_s^+(t)}^b - \frac{dq}{dx} \, dx - \rho(x_s^+, t) \frac{dx_s^+(t)}{dt} \]
\[ = q(a, t) - q(b, t) \]

\[ q(a, t) - q(x_s^-, t) + \rho(x_s^-, t) \frac{dx_s^-(t)}{dt} \]
\[ + q(x_s^+, t) - q(b, t) - \rho(x_s^+, t) \frac{dx_s^+(t)}{dt} \]
\[ = q(a, t) - q(b, t) \]

Now let \( \epsilon \to 0 \) \[ x_s^- \to x_s^+ \]

\[ \frac{dx_s^-(t)}{dt} = \frac{dx_s^+(t)}{dt} = \frac{dx_s(t)}{dt} \]

but \[ \rho(x_s^-, t) \neq \rho(x_s^+, t) \]
\[ q(x_s^-, t) \neq q(x_s^+, t) \]
\[ (\rho^- - \rho^+) \frac{dx_5(t)}{dt} = (\gamma^- - \gamma^+) \] or

\[ \frac{dx_5(t)}{dt} = \frac{(\gamma^- - \gamma^+)}{(\rho^- - \rho^+) \rho} = \frac{(\gamma^+ - \gamma^-)}{(\rho^+ - \rho^-) \rho} = \frac{[g_{ij}]}{[\rho]} \]

Another (less rigorous) argument states that flux in = Flux out across the moving shock

\[ \rho(x_s^-, t) \left[ u(x_s^-, t) - \frac{dx_5(t)}{dt} \right] \leftarrow \text{velocity relative to the moving shock} \]

\[ \rho(x_s^+, t) \left[ u(x_s^+, t) - \frac{dx_5(t)}{dt} \right] \rightarrow \]

\[ \rho^- u^- - \rho^- \frac{dx_5(t)}{dt} = \rho^+ u^+ - \rho^+ \frac{dx_5(t)}{dt} \]

\[ \Rightarrow \frac{dx_5(t)}{dt} = \frac{\rho^+ u^+ - \rho^- u^-}{\rho^+ - \rho^-} = \frac{[g_{ij}]}{[\rho]} \]
The Entropy Condition

\[ x = c(p) t + x_0 \]

Characteristics flow into the shock from both sides

\( c(p) \text{ left} > \text{shock velocity} > c(p) \text{ right} \)

\[ c(p(x_s^-)) > \frac{dx_s}{dt} > c(p(x_s^+)) \]

\( c(p) \) must be a decreasing function of \( p \)
Green light turns red

\[
\frac{d\rho}{dt} + u_{\text{max}} \left(1 - \frac{2\rho}{\rho_{\text{max}}} \right) \frac{d\rho}{dx} = 0 \quad x < 0
\]

with a light at \(x = 0\)

\[
\rho(x, 0) = \rho_0 \quad x < 0
\]

\[
\rho(0, t) = \rho_{\text{max}} \quad t > 0
\]

\[
c(\rho) = u_{\text{max}} \left(1 - \frac{2\rho}{\rho_{\text{max}}} \right) \quad x < 0
\]

\[
c(\rho) = -u_{\text{max}} \quad x = 0
\]

\[
x = -u_{\text{max}} t
\]

\[
x = c(\rho_0) t + x_0 \quad x_0 < 0
\]
So there is a shock that forms

\[ x = -u_{max} t \]

\[ x = c(p_0) t + x_0 \]

\[ x_0 < 0 \]

With \( \frac{dx_5(t)}{dt} = \left[ \begin{array}{c} q \\ \frac{\theta}{\rho} \end{array} \right] \)

\[ q = \rho u \]

\[ = \rho \left( 1 - \frac{\rho}{\rho_{max}} \right) u_{max} \]

So \( \frac{dx_5(t)}{dt} = u_{max} \left( \frac{\left[ \rho \right]}{\left[ \rho \right]_{max}} - \frac{\left[ \rho^2 \right]}{\left[ \rho \right]_{max} \rho_{max}} \right) \)

\[ = u_{max} \left( 1 - \frac{(\rho^2 - \rho_{0}^2)}{(\rho_{max}^2 - \rho_{0}^2)} \right) \]

\[ = u_{max} \left( 1 - \frac{\rho_{0}^2}{\rho_{max}^2} \right) \]
Now with $c^+ = c_{max}$, $c^- = c_0$,

\[ \frac{dx_5(t)}{dt} = u_{max} \left( 1 - \frac{c_{max} + c_0}{c_{max}} \right) \]

\[ = -u_{max} \frac{c_0}{c_{max}} \]

The shock speed is negative if cars pile up behind the light.

\[ x_5(t) = -u_{max} \frac{c_0}{c_{max}} t \]

is the line of stopped cars.
Now let's prove Leibniz's Rule:

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) \, dx = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} \, dx + f(b(t),t) \frac{db(t)}{dt} - f(a(t),t) \frac{da(t)}{dt}
\]

Let \( \phi(t) = \int_{a(t)}^{b(t)} f(x,t) \, dx \)

\( \Delta \phi = \phi(t+\Delta t) - \phi(t) \)

\[
= \int_{a(t+\Delta t)}^{b+\Delta b} f(x,t+\Delta t) \, dx - \int_{a(t+\Delta t)}^{b} f(x,t) \, dx
\]

where \( a(t+\Delta t) = a + \Delta a \)

\( b(t+\Delta t) = b + \Delta b \)
\[ \Delta \Phi = \left\{ \int_{a+\Delta a}^{a} + \int_{a}^{b} + \int_{b}^{b+\Delta b} \right\} f(x, t+\Delta t) \, dx \]

\[ = \int_{a}^{b} f(x, t) \, dx \]

\[ \Delta \Phi = - \int_{a}^{a+\Delta a} f(x, t+\Delta t) \, dx + \int_{b}^{b+\Delta b} f(x, t+\Delta t) \, dx \]

\[ + \int_{a}^{b} \left\{ f(x, t+\Delta t) - f(x, t) \right\} \, dx \]

Now we need to use Mean Value Thm.

For \( f(x) \) continuous on \([a, a+\Delta a]\) and differentiable on \((a, a+\Delta a)\)
\[
\text{area } \int_a^{a+\Delta a} F(x) \, dx = F(\xi) \Delta a
\]

where \( \xi \) is in \((a, a+\Delta a)\)

\[
\Delta \phi = -\Delta a \cdot F(\xi, t+\Delta t) + \Delta b \cdot F(\eta, t+\Delta t)
\]

\[
+ \int_a^b \left\{ \frac{F(x, t+\Delta t) - F(x, t)}{\Delta t} \right\} \, dx
\]

\( \xi \) in \((a, a+\Delta a)\), \( \eta \) in \((b, b+\Delta b)\)

Now divide by \( \Delta t \):

\[
\frac{\Delta \phi}{\Delta t} = -\frac{\Delta a}{\Delta t} \cdot F(\xi, t+\Delta t) + \frac{\Delta b}{\Delta t} \cdot F(\eta, t+\Delta t)
\]

\[
+ \int_a^b \left\{ \frac{F(x, t+\Delta t) - F(x, t)}{\Delta t} \right\} \, dx
\]
Take the limit as $\Delta t \to 0$

$$\frac{d\phi}{dt} = -F(a(t), t) \frac{da(t)}{dt} + F(b(t), t) \frac{db(t)}{dt}$$

$$+ \int_{a(t)}^{b(t)} \frac{dF(x, t)}{dt} \, dx \quad \text{(OED)}$$

where

$$\frac{d\phi(t)}{dt} = \frac{d}{dt} \int_{a(t)}^{b(t)} F(x, t) \, dx$$
Quasi-linear, 1st-order PDEs

What if we add some diffusion to our gas PDE that we know supports shock-like solutions.

Expect smoothed-cut shocks?

\[ \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \]

\[ q(p) = u(p) \rho \quad u(p) = \max \left(1 - \frac{p}{\rho_{\max}}\right) \]

Balance nonlinearity and diffusion

Let's work with a simpler eqn. called Businger's Equation because algebra is simpler.

\[ \frac{\partial u}{\partial t} + \rho \frac{\partial u}{\partial x} = 2 \nu \frac{\partial^2 u}{\partial x^2} \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 2 \nu \frac{\partial^2 u}{\partial x^2} \quad q = \frac{u^2}{2} = q(u) \]

Let's look for special solutions

\[ u = u(x-ct) = u(x) \text{ traveling wave} \]

Solutions with speed \( c_s \), we need to find \( c_s \).
\[-c_0 u'(\xi) + u'(\xi) u'(\xi) = \frac{\gamma}{2} u''(\xi)\]

Integrate w.r.t. \(\xi:\)

\[-c_0 u + \frac{u^2}{2} = \gamma u' + A\]

Now assume there is a jump transition:

\[u = u_1 \quad \xi \to -\infty\]

\[u = u_2 \quad \xi \to +\infty\]

\[-A - c_0 u_1 + \frac{u_1^2}{2} = 0 \quad \xi \to -\infty\]

\[-A - c_0 u_2 + \frac{u_2^2}{2} = 0 \quad \xi \to +\infty\]

\[c_0 = \frac{u_2^2}{2} - \frac{u_1^2}{2} = \frac{\frac{[u]}{L_0}}{u_1 - u_2}\]

Same as the shock velocity!
But this did not distinguish between

\( u_1 < u_2 \) and \( u_1 > u_2 \)

Which one is a shock? \( c(u) \)
It is clear that we need \( u(x,0) \) decreasing with \( x \) for a shock-like solution.

Back to our analysis \( u(x-c_s t) \) satisfies

\[
-C_s u + \frac{u^2}{2} = u' + A
\]

with

\[
C_s = \frac{\int u^2 \, d\gamma}{\int u_1 \, d\gamma} = \frac{\frac{u_1^2}{2} + \frac{u_2^2}{2}}{\frac{u_1 - u_2}{2}} = \frac{1}{2} (u_1 + u_2)
\]

**Algebra:**

\[
A = -C_s u_1 + \frac{u_1^2}{2} = -C_s u_2 + \frac{u_2^2}{2}
\]

\( S \to -\infty \quad S \to +\infty \)

and

\[
u u' = -C_s u + \frac{u^2}{2} - A
\]
\[ u' = -c_1 u + u^2 + c_2 u - u^2 \]

\[ = \gamma F(s, u) \]

\[ \Rightarrow u'(s) = F(s, u) \quad \text{nonlinear 1st order ODE} \]

which can be solved using Method of Direction Fields

If \( F(u) > 0 \) then

\[ u(s) \text{ is increasing} \]

Fixed point

\[ \text{at } s \]

Fixed point

\[ \text{at } s \]
If $F(u) \leq 0$ \hspace{1cm} (2)

Need \( \phi \) for shock-like solutions

\[ \Rightarrow \gamma F(u) \leq 0 = -c_3 u + \frac{u^2}{2} + \frac{c_5}{2} u_1 - u_1^2 \]

\[ = c_5 (u - u) - \frac{1}{2} (u_1^2 - u^2) \]

\[ = \frac{1}{2} (u_1 + u_2) (u_1 - u) - \frac{1}{2} (u_1 - u)(u_1 + u) \]

\[ = \frac{1}{2} (u_1 - u) \left[ u_1 + u_2 - y_y - u \right] \]

\[ = \frac{1}{2} (u_1 - u)(u_2 - u) \]

\[ \leq 0 \hspace{1cm} \text{since } \sqrt{\text{consistent}} \]