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Green's Functions for ODEs :

Standard Form : $y''(x) + a(x)y'(x) + b(x)y(x) = g(x)$

the integrating factor $p(x) = \exp\left[\int a(x)\,dx\right] \Longrightarrow$

SL Form :

$$-\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\}y + q(x)\,y = F(x)$$

$$q(x) = -p(x)b(x) \quad , \quad F(x) = -p(x)g(x)$$

$L\,y(x) = F(x)$ where the linear, differential operator

$$L = -\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$$

has nice symmetry properties (why it is helpful)

The Green's function is defined by

$L\,G(x;a) = \delta(x-a)$ with

* $\delta(x-a) = \delta(a-x)$ symmetric

* $\delta(x-a) = 0$ for $x \neq a$

* $\displaystyle\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\, dx = f(a)$ for $f(x)$ continuous at $x = a$

Then $\displaystyle y(x) = y_h(x) + \int_{-\infty}^{\infty} f(a)\, G(x;a)\, da$

as can be verified by direct substitution:

$$L\left\{ y(x) = y_h(x) + \int_{-\infty}^{\infty} f(a)\, G(x;a)\, da \right\}$$

$$Ly = Ly_h + \int_{-\infty}^{\infty} f(x)\, L\, G(x;a)\, da$$

$$= 0 + \int_{-\infty}^{\infty} f(a)\, \delta(x-a)\, da$$

$$= f(x)$$

Lets remind ourselves about the Dirac delta:

$$\delta(x-a) = 0 \quad x \neq a \quad , \quad \int_{-\infty}^{\infty} f(a)\, \delta(x-a)\, da = f(x)$$

$$\left[\text{with special case} \quad \int_{-\infty}^{\infty} \delta(x-a)\, da = 1 \right]$$

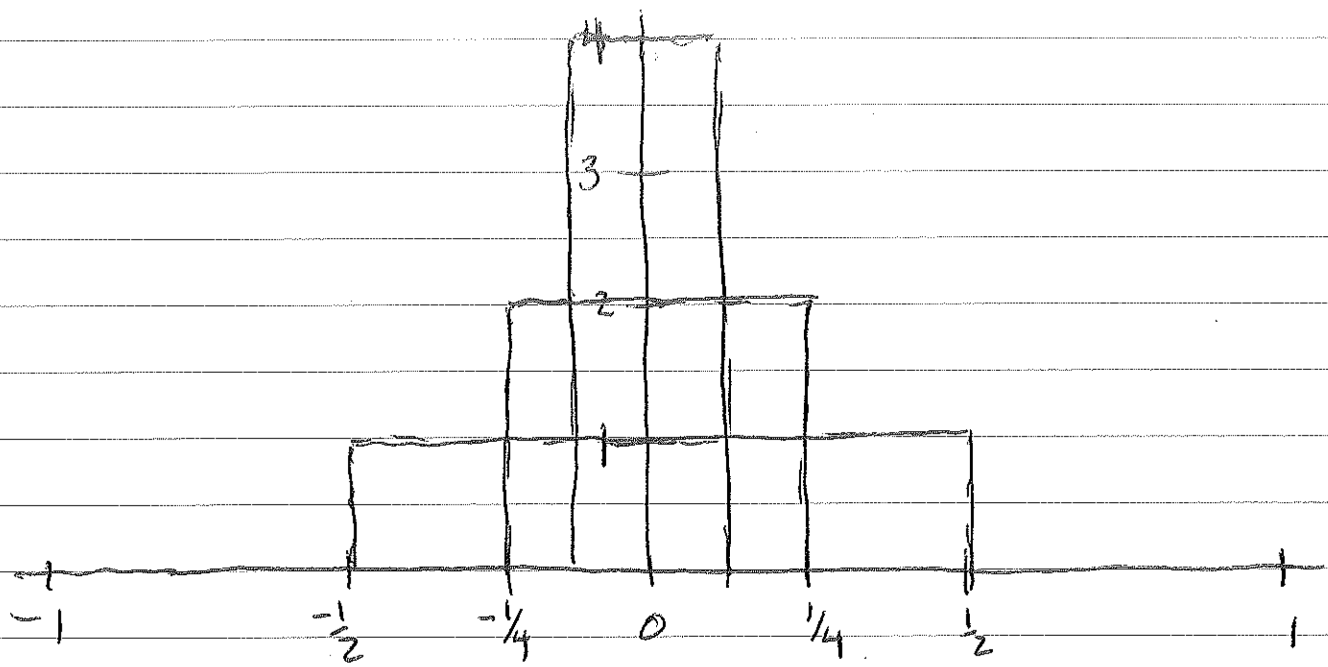
A mathematical idealization of a "unit impulse" (with unit area); represented as the limit of several functions, e.g.

$$F_\varepsilon(x) = \begin{cases} 0 & x < a - \tfrac{1}{2}\varepsilon \\ \frac{1}{\varepsilon} & a - \tfrac{1}{2}\varepsilon \leq x \leq a + \tfrac{1}{2}\varepsilon \\ 0 & x > a + \tfrac{1}{2}\varepsilon \end{cases}$$

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(x) = \delta(x-a)$$

Another: $\displaystyle \lim_{\varepsilon \to 0^+} (\pi \varepsilon)^{-1/2} \exp\left[\frac{-(x-a)^2}{\varepsilon} \right]$

Picture of $F_\varepsilon(x)$ for $a=0$, several values of ε:



Lets use $F_\varepsilon(x)$ to show that

$$\int_{-\infty}^{\infty} \delta(x-a)\, f(x)\, dx = f(a) \quad \text{for } f(x) \text{ continuous at } x=a$$

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} f(x)\, F_\varepsilon(x)\, dx = \lim_{\varepsilon \to 0^+} \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)\, \frac{1}{\varepsilon}\, dx$$

$$= \lim_{\varepsilon \to 0^+} \left. \frac{G(x)}{\varepsilon} \right|_{a-\varepsilon/2}^{a+\varepsilon/2} \quad \text{with } G'(x) = f(x)$$

$$\lim_{\varepsilon \to 0^+} \frac{G(a+\varepsilon/2) - G(a-\varepsilon/2)}{\varepsilon} = G'(a) = f(a)$$

Heaviside Function: (finite jump at $x=a$)

$$H(x-a) = \int_{-\infty}^{x} \delta(t-a)\,dt = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}$$

and it makes sense to define $H(x-a) = \frac{1}{2}$ at $x=a$ by symmetry

Ramp Function:

$$r(x-a) = \int_{-\infty}^{x} H(t-a)\,dt = \begin{cases} 0 & x \leq a \\ x-a & x \geq a \end{cases}$$

(continuous at $x=a$)

Notice that integration smooths:

$\delta(x-a)$ has "infinite discontinuity" at $x=a$

$H(x-a)$ " finite discontinuity at $x=a$

$r(x-a)$ is continuous at $x=a$

Let find $G(x;a)$ for $L = -\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x)$

Assume we can find 2 linearly independent solutions to the homogeneous problem $Ly = 0$:
$y_1(x)$, $y_2(x)$

For $x \neq a$ $\quad LG(x;a) = 0 \quad \Rightarrow$

$$G(x;a) = \begin{cases} A_1 y_1(x) + A_2 y_2(x) & x < a \\ B_1 y_1(x) + B_2 y_2(x) & x > a \end{cases}$$

4 unknowns; Need 4 conditions

2 conditions arise from integrating across the discontinuity:

Definition $\quad LG(x;a) = \delta(x-a) \quad \Rightarrow$

$$\int_{a-\varepsilon}^{a+\varepsilon} \left[-\frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} G(x;a) + q(x)G(x;a) \right] dx$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) = 1$$