Math 322 | 33

We want to solve

$$-\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x)y(x) = F(x) \qquad \alpha < x < \beta$$

with appropriate boundary conditions.

$$L = -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$$

and we know $y(x) = y_h(x) + y_p(x)$

with Green's Function defined by

$$L\,G(x;a) = \delta(x-a), \text{ then}$$

$$y(x) = y_h(x) + \int_\alpha^\beta F(a)\,G(x;a)\,da$$

as can be verified by direct substitution using

* $\delta(x-a) = \delta(a-x)$
* $\delta(x-a) = 0 \qquad x \neq a$
* $\displaystyle\int_{-\infty}^{\infty} \delta(x-a)F(x)\,dx = F(a)$

$$L\,G(x;a) = 0 \qquad x \neq a \qquad \Rightarrow$$

$$G(x;a) = \begin{cases} A_1 y_1(x) + A_2 y_2(x) & x < a \\ B_1 y_1(x) + B_2 y_2(x) & x > a \end{cases}$$

4 unknowns $\Rightarrow$ need 4 conditions: 2 boundary conditions + 2 conditions arising when we integrate across the discontinuity at $x = a$; $L\,G(x;a) = \delta(x-a)$ &

$$\lim_{\varepsilon \to 0^+} \int_{a-\varepsilon}^{a+\varepsilon} \left[-\frac{d}{dx}\left\{ p(x)\frac{dG(x;a)}{dx} \right\} + q(x)\,G(x;a) \right] dx$$

$$= \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) = \underline{1}$$

$\Rightarrow$ 2 derivatives of $G(x;a)$ is "like" $\delta(x-a)$

$\underline{1}$ " " " " " $H(x-a)$

0 " " " is continuous

$\Rightarrow$ $G(x;a)$ is continuous!

After integration

$$\lim_{\varepsilon \to 0^+} \left. -p(x) \frac{dG(x;a)}{dx} \right|_{a-\varepsilon}^{a+\varepsilon} + \left. W(x;a) \right|_{a-\varepsilon}^{a+\varepsilon} = 1$$

where $W(x;a)$ is the anti-derivative of $q(x)G(x;a)$; $W(x;a)$ is continuous since $q(x)$ and $G(x;a)$ are continuous.

So we arrive at 2 conditions :

$$\lim_{\varepsilon \to 0^+} \left. -p(x) \frac{dG(x;a)}{dx} \right|_{a-\varepsilon}^{a+\varepsilon} + 0 = 1$$

$$\lim_{\varepsilon \to 0^+} \left. G(x;a) \right|_{a-\varepsilon}^{a+\varepsilon} = 0 \quad \text{(continuity of } G(x;a)\text{)}$$

These can be rewritten . . .

$$\lim_{\varepsilon \to 0^+} \frac{dG(x;a)}{dx}\Bigg|_{a-\varepsilon}^{a+\varepsilon} = \frac{-1}{p(a)}$$

since $p(x)$ is continuous at $x=a$

$$\lim_{\varepsilon \to 0^+} G(x;a)\Bigg|_{a-\varepsilon}^{a+\varepsilon} = 0$$

The other 2 conditions will come from bundary conditions.

We can now make the connection to Variation of Parameters.

① $B_1 y_1'(a) + B_2 y_2'(a) - A_1 y_1'(a) - A_2 y_2'(a) = -\frac{1}{p(a)}$

② $B_1 y(a) + B_2 y_2(a) - A_1 y_1(a) - A_2 y_2(a) = 0$

$$\Rightarrow \quad B_1 - A_1 = \frac{y_2(a)}{p(a)\, W[y_1(a), y_2(a)]}$$

$$B_2 - A_2 = \frac{-y_2(a)}{p(a)\, W[y_1(a), y_2(a)]}$$

$$W[y_1(a), y_2(a)] = y_1(a) y_2'(a) - y_2(a) y_1'(a)$$

How do we choose b.c.s for $G(x;a)$?

The equation we are trying to solve is

$$L\,y(x) = F(x)$$

Next: Multiply by $G(x;a)$; integrate over the domain; use Green's formula; examine the boundary terms to decide how to choose boundary conditions for $G(x;a)$

$$\int_\alpha^\beta G(x;a)\left\{ L\,y(x) - F(x) \right\} dx = 0$$

integrate by parts twice:

$$\int_\alpha^\beta y(x)\, L\, G(x;a)\, dx$$

$$+ \left[-p(x)\, G(x;a) \frac{dy(x)}{dx} + p(x)\, y(x) \frac{dG(x;a)}{dx} \right]\Bigg|_\alpha^\beta$$

$$- \int_\alpha^\beta G(x;a)\, F(x)\, dx = 0$$

$$\int_{\alpha}^{\beta} y(x)\,\delta(x-a)\,dx = \text{boundary terms}\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} G(x;a)\,F(x)\,dx$$

$$y(a) = y_h(a) + y_p(a)$$

$$y(a) = y_p(a) + \left[p(x) G(x;a) \frac{dy(x)}{dx} - p(x) y(x) \frac{dG(x;a)}{dx} \right] \Bigg|_{\alpha}^{\beta}$$

and we will chose b.c.s for $G(x;a)$ such that the RHS is known

* if $y(\alpha), y(\beta)$ given $\Rightarrow$ choose $G(\alpha;a) = G(\beta;a) = 0$

 can be non-homogeneous

* if $y'(\alpha), y'(\beta)$ given $\Rightarrow$ choose $G'(\alpha;a) = G'(\beta;a) = 0$

* if $y(\alpha), y'(\beta)$ given $\Rightarrow$ $G(\alpha;a) = 0$ $G'(\beta,a) = 0$

* if $y'(\alpha), y(\beta)$ given $\rightarrow$ $G'(\alpha;a) = 0$ $G(\beta,a) = 0$

etc. works even for Neuton-type b.cos!

Chose the homogeneous version of whatever boundary conditions are given for $y(x)$, $y'(x)$ at α, β.

* By symmetry of the operator in SL form $\Rightarrow$
$G(x;a)$ will be symmetric (any other choice breaks the symmetry in general)

* What we really want is an expression for $y(x)$, so switch the roles of x, a

$Ly(a) = F(a)$ $\quad$ L an operator wrt a $\Rightarrow$

$$\int_{\alpha}^{\beta} G(a;x)\left\{ Ly(a) - F(a) \right\} da = 0 \Rightarrow$$

$$y(x) = \int_{\alpha}^{\beta} G(a;x)\,F(a)\,dx$$

$$+ \left[p(a)\,G(a;x)\frac{dy(a)}{da} - p(a)\,y(a)\frac{dG(a;x)}{da} \right]\Bigg|_{a=\alpha}^{a=\beta}$$

$$= y_p(x) + y_h(x)$$