$$L\,y(x) = f(x) \qquad \alpha < x < \beta$$

with possibly non-homogeneous boundary conditions

Define the Green's Function by

$$L\,G(x;a) = \delta(x-a)$$

* Integrating: $\displaystyle \lim_{\varepsilon \to 0^+} \int_{a-\varepsilon}^{a+\varepsilon} L\,G(x;a)\,dx = \lim_{\varepsilon \to 0^+} \int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a)\,dx$

$$\Longrightarrow \quad \lim_{\varepsilon \to 0^+} \frac{dG(x;a)}{dx}\bigg|_{a-\varepsilon}^{a+\varepsilon} = \frac{-1}{p(a)}$$

$$\lim_{\varepsilon \to 0^+} G(x;a)\bigg|_{a-\varepsilon}^{a+\varepsilon} = 0$$

* Now multiply $G(x;a)\{L\,y(x) = f(x)\}$ and integrate

or better $G(a;x)\{L\,y(a) = f(a)\}$ and integrate

over the whole domain

$$\int_{\alpha}^{\beta} G(a;x)\left\{ L\,y(a) - F(a) \right\} da = 0 \qquad \Longrightarrow$$

$$y(x) = \int_{\alpha}^{\beta} G(a;x)\, F(a)\, dx$$

$$+ \left[p(a)\, G(a;x)\, \frac{dy(a)}{da} - p(a)\, y(a)\, \frac{dG(a;x)}{da} \right] \Bigg|_{a=\alpha}^{a=\beta}$$

$$= y_p(x) + y_h(x)$$

This formula tells us what boundary conditions to choose for $G(a;x)$ $\{$ or $G(x;a$ $\}$

Choose the homogeneous version

of the boundary conditions that are

given for $y(x)$, $\dfrac{dy(x)}{dx}$

$\boxed{\text{Simple Example \#1}}$

$$y'' = f(x) \quad 0 < x < 1 \quad y(0) = 0 \quad y(1) = \gamma$$

In SL form: $-y'' = -f(x)$, $p(x) = 1$, $q(x) = 0$

$$G(x;a) = \begin{cases} A_1 x + A_2 & x < a \\ B_1 x + B_2 & x > a \end{cases}$$

Use boundary conditions $G(0;a) = 0$, $G(1;a) = 0$:

$$G(0;a) = 0 \implies A_1 \cdot 0 + A_2 = 0 \implies \boxed{A_2 = 0}$$

$$G(1;a) = 0 \implies B_1 \cdot 1 + B_2 = 0 \implies \boxed{B_1 + B_2 = 0}$$

$$\lim_{\varepsilon \to 0^+} G(x;a)\Big|_{a-\varepsilon}^{a+\varepsilon} = 0 \implies$$

$$\boxed{B_1 a + B_2 - A_1 a = 0}$$

$$\lim_{\varepsilon \to 0^+} \frac{dG(x;a)}{dx}\Bigg|_{a-\varepsilon}^{a+\varepsilon} = -1 \implies$$

$$\boxed{B_1 - A_1 = -1}$$

4 equations, 4 unknowns $\implies$

$$G(x;a) = \begin{cases} (1-a)\,x & x<a \\ (1-x)\,a & x>a \end{cases}$$ Symmetric

$$G(a;x) = \begin{cases} (1-x)\,a & a<x \\ (1-a)\,x & a>x \end{cases}$$

$$\frac{dG(x;a)}{dx} = \begin{cases} (1-a) & x<a \\ -a & x>a \end{cases}$$

$$\frac{dG(a;x)}{da} = \begin{cases} (1-x) & a<x \\ -x & a>x \end{cases}$$

$$y(x) = \int_0^1 G(a;x)\,[-f(a)]\,da$$

$$+ \left[\cancel{p(a)\,G(a;x)}^{\,0}\,\frac{dy(a)}{da} - p(a)\,y(a)\,\frac{dG(a;x)}{da} \right] \Bigg|_{a=0}^{a=1}$$

$$= \int_0^x (1-x)a\,(-F(a))\,da + \int_x^1 (1-a)\,x\,(-F(a))\,da$$

$$- \gamma(-x)$$

$$= \int_0^x (x-1)\,a\,F(a)\,da + \int_x^1 (a-1)\,x\,F(a)\,da + \gamma x$$

Check for $F(x) = x^2$ $\quad F(a) = a^2$

$$= \int_0^x x\,a^3\,da - \int_0^x a^3\,da + \int_x^1 a^3 x\,da$$

$$- \int_x^1 x\,a^2\,da + \gamma x$$

$$= x\frac{a^4}{4}\Big|_0^x - \frac{a^4}{4}\Big|_0^x + x\frac{a^4}{4}\Big|_x^1 - \frac{x a^3}{3}\Big|_x^1 + \gamma x$$

$$= \cancel{\frac{x^5}{4}} - 0 - \frac{x^4}{4} + 0 + \frac{x}{4} - \cancel{\frac{x^5}{4}} - \frac{x}{3} + \frac{x^4}{3} + \gamma x$$

$$= \frac{x^4}{12} - \frac{x}{12} + \gamma x$$

Does it satisfy the equation and boundary conditions?

Suppose we now have Poisson's Eqn:

$$\nabla^2 u(\underline{x}) = f(\underline{x})$$

or the Heat Eqn. with sorces/sinks

$$\frac{\partial u(\underline{x},t)}{\partial t} = K\nabla^2 u(\underline{x},t) + Q(\underline{x},t)$$

in 2D or 3D. Can we use a Green's Function approach? Yes!

Poisson's Eqn. is the most natural extension of the ODE formulation

We want to solve $\nabla^2 u(\underline{x}) = f(\underline{x})$ with possibly non-homogeneous boundary conditions

Define $$\nabla^2 G(\underline{x}\,;\,\underline{x}_0) = \delta(\underline{x} - \underline{x}_0)$$

Note that the Green's Function is the electrostatic potential for a point charge at $\underline{x} = \underline{x}_0$